Name: ________________________________

Pid: ________________________________

Note that every statement in the midterm should be proved. The only exceptions are statements that were proven in previous homework or midterms and statements proven earlier in the class.

1. (50 points) Tick if the answer for the question is yes (this is the only question where you do not need to prove correctness of your answer).
   - Monty and a goat have apples and bananas. Monty likes apples and dislikes bananas (the more bananas he has, the worse off he is), and the goat likes bananas and dislikes apples. There are 100 apples and 100 bananas available. Is it true that the only Pareto optimal way to split fruits is to give to Monty all the apples and to the goat all the bananas?
   - Is \((a, b)\) a Nash equilibrium in the following game?
     \[
     \begin{array}{ccc}
     \text{a} & \text{b} & \text{c} \\
     \hline
     \text{a} & 1, 1 & 2, 1 & 5, 1 \\
     \text{b} & 0, 1 & 0, 2 & 5, 0 \\
     \text{c} & 4, 1 & 1, -1 & 7, 2 \\
     \end{array}
     \]
   - Is \((c, a)\) Pareto optimal in the following game?
     \[
     \begin{array}{ccc}
     \text{a} & \text{b} & \text{c} \\
     \hline
     \text{a} & 2, 1 & 2, 3 & 5, 2 \\
     \text{b} & 0, 3 & 5, 10 & 1, 1 \\
     \text{c} & 10, 4 & 2, 7 & 3, 4 \\
     \end{array}
     \]
   - Ian and Masha are playing a game. In this game the starting configuration is a row of coins showing heads. The two players alternate; each player, on his or her turn, flips one coin from heads to tails and may not flip a coin next to a coin showing tails. Is 2 the value of the Grundy function of this game with 6 coins?
   - Lloyd and Dunne play the following game.
     \[
     \begin{array}{cc}
     \text{a} & \text{b} \\
     \hline
     \text{a} & 1, 1 \quad 2, 3 \\
     \text{b} & 0, 1 \quad 4, 2 \\
     \end{array}
     \]
     Lloyd plays the strategy \(a\) with probability \(1/3\) and the strategy \(b\) with probability \(2/3\). Dunne plays the strategy \(a\) with probability \(1/4\) and the strategy \(b\) with probability \(3/4\). Is 2 the average gain of Lloyd in this case?
2. (10 points) Alice and Bob play a game with a regular polygon. Players alternate; each player, on his or
her turn, draws one diagonal (a line connecting two not adjacent vertices) or draws two diagonals from
the same vertex to two adjacent vertices such that none of the drawn diagonals intersect each other. A
player loses when he or she has no legal move.

(a) (10 points) Who wins in this game if the initial polygon has $n$ vertices.

**Solution:** The first player can always win in this game. Consider two following cases.

- If number $n$ of vertices in the polygon is even we can split our polygon into two polygons
  with $\frac{n+2}{2}$ vertices, respectively. This will split the game into the sum of two equal games
  with equal initial states. Hence, we reached a P-position and it implies that the first
  player wins.

- In the case when $n$ is odd we split the polygon into two polygons with $\frac{n+1}{2}$ vertices. Again
  it is a sum of two equal games with equal initial states and the first player wins.
(b) (5 points) Prove that the Grundy function of this game is periodic starting from some point, i.e. there are \( k, l \in \mathbb{N} \) such that \( g(n + l) = g(n) \) for \( n \geq k \).

You can use the fact that for \( g(n + 12) = g(n) \) for \( 300 > n \geq 100 \).

**Solution:** We prove that \( g(n - 12) = g(n) \) for \( n \geq 112 \). To prove this statement we use strong induction by \( n \). The statement is true for \( n < 300 \) by the statement of the problem.

Let us now prove the induction step for \( n \geq 300 \). Note that we can split a polygon with \( n \) vertices into two polygons with \( x \geq 3 \) and \( y \geq 3 \) vertices iff \( x + y = n + 2 \) or \( x + y = n + 1 \). Let \( S = \{ g(x) \oplus g(y) : x, y \geq 3 \text{ and } x + y \in \{ n + 1, n + 2 \} \} \) and \( S' = \{ g(x) \oplus g(y) : x, y \geq 3 \text{ and } x + y \in \{ n - 12 + 1, n - 12 + 2 \} \} \).

First of all we prove that \( S \subseteq S' \). Let us consider some move from a polygon with \( n \) to into two polygons with \( x \geq 3 \) and \( y \geq 3 \). Without loss of generality \( x \geq y \) (as a result \( x \geq 150 \)). Note that there is a move from a polygon with \( n - 12 \) to into two polygons with \( x - 12 \) and \( y \) and by the induction hypothesis \( g(x) = g(x - 12) \); hence, \( (g(x) \oplus g(y)) \in S' \).

Secondly, we prove that \( S' \subseteq S \). Let us consider some move from a polygon with \( n - 12 \) to into two polygons with \( x' \geq 3 \) and \( y' \geq 3 \). Without loss of generality \( x' \leq y' \) (as a result \( x' \leq 150 \)). Note that there is a move from a polygon with \( n \) into two polygons with \( x' + 12 \) and \( y' \) and by the induction hypothesis \( g(x') = g(x' + 12) \); hence, \( (g(x') \oplus g(y')) \in S \).

As a result, \( S = S' \) and \( \text{mex}(S) = \text{mex}(S') \) i.e. \( g(n) = g(n - 12) \).
3. (10 points) Eddie and Lana play a game where they each simultaneously announce an integer between 1 and 4 (inclusive). Let \( x \) be the number chosen by Eddie, and let \( y \) be the number chosen by Lana. If \( x \leq y \), then Eddie wins. Otherwise, Lana wins. The losing player pays \( xy \) (i.e. the product of the two numbers) to the winning player. Construct the payoff matrix, and then find a Pareto optimal pairs of strategies (or prove that they do not exist).

Solution: The payoff matrix of this game is

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, -1)</td>
<td>(2, -2)</td>
<td>(3, -3)</td>
<td>(4, -4)</td>
</tr>
<tr>
<td>2</td>
<td>(-2, 2)</td>
<td>(4, -4)</td>
<td>(6, -6)</td>
<td>(8, -8)</td>
</tr>
<tr>
<td>3</td>
<td>(-3, 3)</td>
<td>(-6, 6)</td>
<td>(9, -9)</td>
<td>(12, -12)</td>
</tr>
<tr>
<td>4</td>
<td>(-4, 4)</td>
<td>(-8, 8)</td>
<td>(-12, 12)</td>
<td>(16, -16)</td>
</tr>
</tbody>
</table>

In this game if one player get \( x \) another get \(-x\). Hence, any pair \((x, -x)\) is a Pareto optimal situation (since increasing the payoff of one player decrease the payoff of another) i.e. any pair of strategies is Pareto optimal in this game.