Vector Equations

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We considered the system

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
3 & -7 & 8 & -5 & 8 & 9 \\
3 & -9 & 12 & -9 & 6 & 15
\end{bmatrix}.
\]
We considered the system

$$\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
3 & -7 & 8 & -5 & 8 & 9 \\
3 & -9 & 12 & -9 & 6 & 15
\end{bmatrix}.$$ 

We transformed it into the matrix

$$\begin{bmatrix}
1 & 0 & -2 & 3 & 0 & -5 \\
0 & 1 & -2 & 0 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}.$$
We transformed it into the matrix

\[
\begin{bmatrix}
1 & 0 & -2 & 3 & 0 & -5 \\
0 & 1 & -2 & 0 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

With the corresponding system

\[
\begin{align*}
x_1 - 2x_3 + 3x_4 &= -24 \\
x_2 - 2x_3 &= -7 \\
x_5 &= 4
\end{align*}
\]
Previously On Math 18

With the corresponding system

\[
\begin{align*}
x_1 &- 2x_3 + 3x_4 = -24 \\
x_2 &- 2x_3 = -7 \\
x_5 & = 4
\end{align*}
\]

In other words, the solution set may be described in the following manner.

\[
\begin{align*}
x_1 & = -24 + 2x_3 - 3x_4 \\
x_2 & = -7 + 2x_3 \\
x_5 & = 4 \\
x_3 & \text{ is free} \\
x_4 & \text{ is free}
\end{align*}
\]
Existence and Uniqueness of a Solution

THEOREM

- A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column.
Existence and Uniqueness of a Solution

THEOREM

- A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column.
- If a linear system is consistent, then the solution set contains only unique solution when there are no free variables, or infinitely many solutions when there is at least one free variable.
Existence of a Solution

Let us consider the matrix

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
2 & 1 & 0 & 3 \\
3 & 3 & 1 & 4 \\
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
2 & 1 & 0 & 3 \\
3 & 3 & 1 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
2 & 1 & 0 & 3 \\
3 & 3 & 1 & 4
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
2 & 1 & 0 & 3 \\
3 & 3 & 1 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
3 & 3 & 1 & 4 \\
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
3 & 3 & 1 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & -3 & -2 & -8 \\
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & -3 & -2 & -8 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & -3 & -2 & -8 \\
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & -3 & -2 & -8 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & 0 & 0 & 9 \\
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & 0 & 0 & 9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & 0 & 0 & 9 \\
\end{bmatrix}
\]
Existence of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & -3 & -2 & -5 \\
0 & 0 & 0 & 9 \\
\end{bmatrix}
\]

The system of equation is inconsistent.
Uniqueness of a Solution

Let us consider the matrix

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
2 & 1 & 0 & 1 & 3 \\
3 & 3 & 2 & 1 & 4
\end{bmatrix}
\]
Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
2 & 1 & 0 & 1 & 3 \\
3 & 3 & 2 & 1 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
2 & 1 & 0 & 1 & 3 \\
3 & 3 & 2 & 1 & 4 \\
\end{bmatrix}
\]
### Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
2 & 1 & 0 & 1 & 3 \\
3 & 3 & 2 & 1 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
3 & 3 & 2 & 1 & 4 \\
\end{bmatrix}
\]
Uniqueness of a Solution

$$\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
3 & 3 & 2 & 1 & 4
\end{bmatrix}$$

$$\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & -3 & -1 & -2 & -8
\end{bmatrix}$$
Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & -3 & -1 & -2 & -8 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & -3 & -1 & -2 & -8 \\
\end{bmatrix}
\]
**Uniqueness of a Solution**

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & -3 & -1 & -2 & -8
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & 0 & 1 & -1 & -3
\end{bmatrix}
\]
Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & 0 & 1 & -1 & -3 \\
0 & 0 & 1 & -1 & -3
\end{bmatrix}
\]
Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
0 & -3 & -2 & -1 & -5 \\
0 & 0 & 1 & -1 & -3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & -3 & 0 & -3 & -11 \\
0 & 0 & 1 & -1 & -3 \\
\end{bmatrix}
\]
Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & -3 & 0 & -3 & -11 \\
0 & 0 & 1 & -1 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -2 & -19/3 \\
0 & 1 & 0 & 1 & 11/3 \\
0 & 0 & 1 & -1 & -3
\end{bmatrix}
\]
Uniqueness of a Solution

\[
\begin{bmatrix}
1 & 0 & 0 & -2 & -19/3 \\
0 & 1 & 0 & 1 & 11/3 \\
0 & 0 & 1 & -1 & -3
\end{bmatrix}
\]

\[
x_1 = -19/3 + 2x_4 \\
x_2 = 11/3 - x_4 \\
x_3 = -3 + x_4 \\
x_4 \text{ is free}
\]
Uniqueness of a Solution

\[ x_1 = -\frac{19}{3} + 2x_4 \]
\[ x_2 = \frac{11}{3} - x_4 \]
\[ x_3 = -3 + x_4 \]
\[ x_4 \text{ is free} \]

\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 4 \\
2 & 1 & 0 & 1 & 3 \\
3 & 3 & 2 & 1 & 4 \\
\end{bmatrix}
\]
Vectors in \( \mathbb{R}^2 \)

**DEFINITION**

A matrix with only one column is called a **vector** or **column vector**.
Vectors in $\mathbb{R}^2$

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A matrix with only one column is called a **vector** or **column vector**.

**EXAMPLE**

\[ u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ .2 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \]

where $w_1$ and $w_2$ are any real numbers.
Vectors in $\mathbb{R}^2$

DEFINITION

A matrix with only one column is called a vector or column vector.

EXAMPLE

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} .1 \\ .2 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where $w_1$ and $w_2$ are any real numbers.

DEFINITION

The set of all vectors with two entries is called $\mathbb{R}^2$. 
DEFINITIONS

Two vectors are equal iff their corresponding entries are equal.
Equal Vectors

DEFINITIONS

Two vectors are equal iff their corresponding entries are equal.

EXAMPLE

Are vectors \[
\begin{bmatrix}
7 \\
2
\end{bmatrix}
\] and \[
\begin{bmatrix}
2 \\
7
\end{bmatrix}
\] equal?
Equal Vectors

DEFINITIONS

Two vectors are equal iff their corresponding entries are equal.

EXAMPLE

Vectors \([7, 2]\) and \([2, 7]\) are not equal.
**Equal Vectors**

**DEFINITIONS**

Two vectors are equal iff their corresponding entries are equal.

**EXAMPLE**

Are vectors \[
\begin{bmatrix}
44 \\
1
\end{bmatrix}
\quad \text{and} \\
\begin{bmatrix}
2 \cdot 22 \\
2 - 1
\end{bmatrix}
\] equal?
Equal Vectors

DEFINITIONS

Two vectors are equal iff their corresponding entries are equal.

EXAMPLE

Vectors \[\begin{bmatrix} 44 \\ 1 \end{bmatrix}\] and \[\begin{bmatrix} 2 \cdot 22 \\ 2 - 1 \end{bmatrix}\] are equal.
Sum of Vectors

**DEFINITION**

Given two vectors $u$ and $v$ their **sum** is obtained by adding corresponding entries of $u$ and $v$. 

\[
\begin{bmatrix}
2 \\
4
\end{bmatrix} + 
\begin{bmatrix}
7 \\
1
\end{bmatrix} = 
\begin{bmatrix}
2 + 7 \\
4 + 1
\end{bmatrix} = 
\begin{bmatrix}
9 \\
5
\end{bmatrix}
\]
Sum of Vectors

**DEFINITION**

Given two vectors $u$ and $v$ their **sum** is obtained by adding corresponding entries of $u$ and $v$.

**EXAMPLE**

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix} =$$
Sum of Vectors

**DEFINITION**

Given two vectors $u$ and $v$ their sum is obtained by adding corresponding entries of $u$ and $v$.

**EXAMPLE**

\[
\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + 7 \\ 4 + 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}
\]
Scalar Multiplies of Vectors

**DEFINITION**

Given a vector $u$ and a real number $c$, the **scalar multiple** of $u$ by $c$ is a vector $cu$ obtained by multiplying each entry of $u$ by $c$. 

**EXAMPLE**

If $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $c = 5$, then $cu = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$. 

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Scalar Multiplies of Vectors

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**EXAMPLE**

if $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $c = 5$, then $cu =$
Scalar Multiplies of Vectors

**DEFINITION**

Given a vector $u$ and a real number $c$, the **scalar multiple** of $u$ by $c$ is a vector $cu$ obtained by multiplying each entry of $u$ by $c$.

**EXAMPLE**

If $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $c = 5$, then $cu = \begin{bmatrix} 5 \cdot 2 \\ 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$. 
Vectors in $\mathbb{R}^n$

**DEFINITION**

If $n > 0$, then $\mathbb{R}^n$ denotes a set of all vectors with $n$ entries. We denote by $0$ the vector consisting of all zeros. Equality, addition and multiplication by scalar we define entry by entry as in $\mathbb{R}^2$. 
Vectors in $\mathbb{R}^n$

**DEFINITION**

If $n > 0$, then $\mathbb{R}^n$ denotes a set of all vectors with $n$ entries. We denote by $0$ the vector consisting of all zeros. Equality, addition and multiplication by scalar we define entry by entry as in $\mathbb{R}^2$.

**ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$**

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = 0 + u = u$
4. $u + (−u) = −u + u = 0$ where $−u$ denotes $(−1)u$
5. $c(u + v) = cu + cv$
6. $(c + d)u = cu + du$
7. $c(du) = (cd)u$
8. $1u = u$
Linear combinations

DEFINITION

Given vectors $u_1, \ldots, u_l \in \mathbb{R}^n$ and real numbers $c_1, \ldots, c_l$. The vector equal to

$$v = c_1 u_1 + \cdots + c_l u_l$$

is called a **linear combination** of $u_1, \ldots, u_l$. 
Linear combinations

Let us consider vectors

\[ a = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \]

Does \( c \) can be written as a linear combination of \( a \) and \( b \)?
Linear combinations

Let us consider vectors

\[
\begin{align*}
  a &= \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \\
  b &= \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \\
  c &= \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.
\end{align*}
\]

Are there real \(x\) and \(y\) such that \(xa + yb = c\).
Linear combinations

Let us consider vectors

\[ a = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \]

Are there real \( x \) and \( y \) such that \( xa + yb = c \).

\[
\begin{bmatrix}
  x + 2y \\
  -2x + 5y \\
  -5x + 6y
\end{bmatrix}
= \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}
\]
Linear combinations

Let us consider vectors

\[ a = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and} \quad c = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \]

Are there real \( x \) and \( y \) such that

\[
\begin{align*}
x + 2y &= 7 \\
-2x + 5y &= 4 \\
-5x + 6y &= -3.
\end{align*}
\]
Vector equations

THEOREM

A vector equation

\[ x_1 a_1 + \ldots + x_l a_l = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & \ldots & a_l & b \\
\end{bmatrix}
\]
Vector equations

**THEOREM**

A vector equation

\[ x_1 a_1 + \ldots x_l a_l = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
  a_1 & a_2 & \ldots & a_l & b
\end{bmatrix}
\]

**DEFINITION**

If \( v_1, \ldots, v_l \in \mathbb{R}^n \), then a set of all linear combinations of them is called \( \text{Span} \{ v_1, \ldots, v_l \} \).
Vector equations

**DEFINITION**

If $v_1, \ldots, v_l \in \mathbb{R}^n$, then a set of all linear combinations of them is called $\text{Span}\{v_1, \ldots, v_l\}$.

**REMARK**

Asking if $b \in \text{Span}\{a_1, \ldots, a_l\}$ is equivalent to asking if an equation $x_1a_1 + \cdots + x_la_l = b$ has a solution.
A Geometric Description of Span

1. If \( u \in \mathbb{R}^2 \) or \( u \in \mathbb{R}^3 \), then \( \text{Span} \{ u \} \) represents a line.
A Geometric Description of Span

1. If $u \in \mathbb{R}^2$ or $u \in \mathbb{R}^3$, then $\text{Span} \{ u \}$ represents a line.
2. If $u, v \in \mathbb{R}^3$ and $v$ is not multiple of $u$, then $\text{Span} \{ u, v \}$ represents a plane.
EXAMPLE

A company manufactures two products. For 1$ worth of product B, the company spends .45 on materials, .25 on labor, and .15 on overhead. For 1$ worth of product C, the company spends .40 on materials, .30 on labor, and .20 on overhead.
Linear Combinations in Applications

EXAMPLE

A company manufactures two products. For 1$ worth of product B, the company spends .45 on materials, .25 on labor, and .15 on overhead. For 1$ worth of product C, the company spends .40 on materials, .30 on labor, and .20 on overhead.

1. What economic interpretation can be given to $100b$ where $b = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix}$?
Linear Combinations in Applications

EXAMPLE

A company manufactures two products. For 1$ worth of product B, the company spends .45 on materials, .25 on labor, and .15 on overhead. For 1$ worth of product C, the company spends .40 on materials, .30 on labor, and .20 on overhead.

① What economic interpretation can be given to $100b where $b = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix}$?

② Suppose the company wishes to manufacture $x_1$ dollars worth of product B and $x_2$ dollars worth of product C. Give a vector that describes the various costs the company will have.
Linear Combinations in Applications

EXAMPLE

A company manufactures two products. For 1$ worth of product B, the company spends .45 on materials, .25 on labor, and .15 on overhead. For 1$ worth of product C, the company spends .40 on materials, .30 on labor, and .20 on overhead.

1. What economic interpretation can be given to $100b$ where $b = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix}$?

2. Suppose the company wishes to manufacture $x_1$ dollars worth of product B and $x_2$ dollars worth of product C. Give a vector that describes the various costs the company will have.

A vector $100b$ lists the various costs for producing the product B 100$ worth.
EXAMPLE

A company manufactures two products. For 1$ worth of product B, the company spends .45 on materials, .25 on labor, and .15 on overhead. For 1$ worth of product C, the company spends .40 on materials, .30 on labor, and .20 on overhead.

1. What economic interpretation can be given to 100b where \( b = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} \)?

2. Suppose the company wishes to manufacture \( x_1 \) dollars worth of product B and \( x_2 \) dollars worth of product C. Give a vector that describes the various costs the company will have.

A vector 100b lists the various costs for producing the product B 100$ worth. The costs of manufacturing \( x_1 \) dollars worth of B are given by vector \( x_1 b \) and the costs of manufacturing \( x_2 \) dollars worth of A are given by vector \( x_2 c \). Hence the total costs are \( x_1 b + x_2 c \).
A Product of a Matrix and a Column

**DEFINITION**

If $A$ is a $n \times m$ matrix, with columns $a_1, \ldots, a_m$ and $x \in \mathbb{R}^n$, then the product of $A$ and $x$ denoted as $Ax$ is

$$Ax = x_1a_1 + \cdots + x_na_n.$$
A Product of a Matrix and a Column

**DEFINITION**

If $A$ is a $n \times m$ matrix, with columns $a_1, \ldots, a_m$ and $x \in \mathbb{R}^n$, then the product of $A$ and $x$ denoted as $Ax$ is

$$Ax = x_1a_1 + \cdots + x_na_n.$$ 

**EXAMPLE**

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
THEOREM

If A is a matrix with columns $a_1, \ldots, a_n \in \mathbb{R}^m$, then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1 a_1 + \cdots + x_n a_n = b$$

which in turn has the same solution as a system of linear equations with augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}.$$
Existence of Solutions

THEOREM

The equation $Ax = b$ has a solution iff $b$ is a linear combination of the columns of $A$. 