Matrix operations
The Matrix of a Linear Transformation

Let us consider columns $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. 
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Suppose $T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \\ 1 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, and $T(e_3) = \begin{bmatrix} -3 \\ 8 \\ 0 \\ 7 \end{bmatrix}$.

Find a formula for image of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. 
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Suppose \( T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}, T(e_2) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \) and \( T(e_3) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} \).

Find a formula for image of \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \).

Note that
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3.
\]
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$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3.$

Since $T$ is linear

$T(x) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) = \begin{bmatrix} 5 \\ -7 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 8 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -3 \\ -7 & 2 & 8 \\ 2 & 1 & 0 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix $A$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$
Existence and Uniqueness Questions

**DEFINITION**

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $b \in \mathbb{R}^m$ is the image of at least one $x \in \mathbb{R}^n$. 
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DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $b \in \mathbb{R}^m$ is an image of *at most* one $x \in \mathbb{R}^n$. 
Uniqueness Question

THEOREM

Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation. Then \( T \) is one-to-one iff the equation \( T(x) = 0 \) has only one solution.
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PROOF.

Since $T$ is linear, $T(0) = 0$ and we know that $T(x) = 0$ has at most one solution and hence only the trivial solution.
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If $T$ is not one-to-one, then there are a $b \in \mathbb{R}^m$ and $u \neq v \in \mathbb{R}^n$ such that $T(u) = T(v) = b$. 
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But since $T$ is linear $T(u - v) = T(u) - T(v) = 0$. But $u - v \neq 0$ hence $T(x) = 0$ has more than one solution, contradiction.
Existence Question

THEOREM

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then:

1. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of $A$ spans $\mathbb{R}^m$;
2. $T$ is one-to-one iff the columns of $A$ are linearly independent.
Existence Question

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PROOF.

We prove earlier that columns of $A$ span $\mathbb{R}^m$ iff for each $b \in \mathbb{R}^m$ the equation $Ax = b$ is consistent i.e. $T(x) = b$ has solution. This is true iff $T$ is $\mathbb{R}^n$ onto $\mathbb{R}^m$. □
THEOREM

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2. $T$ is one-to-one iff the columns of $A$ are linearly independent.

PROOF.

$T(x)$ is one-to-one iff $Ax = 0$ has only trivial solution. This is true only if columns of $A$ are linearly independent.
Special Matrices

- The scalar entry in the \( i \)th row and \( j \)th column of \( A \) is denoted \( a_{ij} \) and is called the \((i,j)\)-entry if \( A \).
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- A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero. Note that identity matrix is diagonal matrix.
- A **zero matrix** is $m \times n$ matrix with all entries are equal to zero.
Sums

**DEFINITION**

If $A$ and $B$ are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sum of the corresponding columns of $A$ and $B$. 

**EXAMPLE**

Let $A = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$. 
Sums

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EXAMPLE

Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 5 & 1 & 5 \\ 2 & 8 & 9 \end{bmatrix}$.
Scalar Multiples

**DEFINITION**

If $r$ is a scalar and $A$ is a matrix, then the scalar multiple $rA$ is the matrix whose columns are $r$ times the corresponding columns in $A$.

**EXAMPLE**

Let $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$. Then $2B = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$. 
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\[ A \cdot (B \cdot x) \]
Matrix Multiplication

\[ x \rightarrow Bx \rightarrow A(Bx) \]
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Matrix Multiplication

Let $A$ be a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $x \in \mathbb{R}^p$. 

DEFINITION
If $A$ is a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $b_1, \ldots, b_p$ are columns of $B$, then the matrix $AB = \begin{bmatrix} Ab_1; \ldots; Ab_p \end{bmatrix}$ is a matrix of size $m \times p$. 

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Matrix Multiplication

Let $A$ be a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $x \in \mathbb{R}^p$. Denote as $b_1$, ..., $b_p$ columns of $B$ and $x_1$, ..., $x_p$ entries of $x$. 
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Let $A$ be a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $x \in \mathbb{R}^p$. Denote as $b_1, \ldots, b_p$ columns of $B$ and $x_1, \ldots, x_p$ entries of $x$. Note that $Bx = x_1b_1 + \ldots x_pb_p$. 
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Let $A$ be a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $x \in \mathbb{R}^p$. Denote as $b_1, \ldots, b_p$ columns of $B$ and $x_1, \ldots, x_p$ entries of $x$. Note that $Bx = x_1 b_1 + \ldots + x_p b_p$. By linearity of multiplication $A(Bx) = x_1 Ab_1 + \cdots + x_p Ab_p$. 
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**DEFINITION**

If is a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $b_1, \ldots, b_p$ are columns of $B$, then the matrix $AB = [Ab_1, \ldots, Ab_p]$ is a matrix of size $m \times p$. 
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**EXAMPLE**

Let us compute $AB$ where

\[
A = \begin{bmatrix}
2 & 3 \\
1 & 5
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
4 & 3 & 6 \\
1 & 2 & 3
\end{bmatrix}
\]

Then,

\[
Ab_1 = \begin{bmatrix}
2 & 3 & 1 \\
5 & & 
\end{bmatrix}
\begin{bmatrix}
4 \\
1
\end{bmatrix} = \begin{bmatrix}
11 \\
1
\end{bmatrix}
\]

\[
Ab_2 = \begin{bmatrix}
2 & 3 & 1 \\
5 & & 
\end{bmatrix}
\begin{bmatrix}
3 \\
2
\end{bmatrix} = \begin{bmatrix}
10 \\
13
\end{bmatrix}
\]

\[
Ab_3 = \begin{bmatrix}
2 & 3 & 1 \\
5 & & 
\end{bmatrix}
\begin{bmatrix}
6 \\
3
\end{bmatrix} = \begin{bmatrix}
21 \\
9
\end{bmatrix}
\]
Matrix Multiplication

DEFINITION

If is a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $b_1, \ldots, b_p$ are columns of $B$, then the matrix $AB = [Ab_1, \ldots, Ab_p]$ is a matrix of size $m \times p$.

EXAMPLE

Let us compute $AB$ where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.
Matrix Multiplication

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If is a $m \times n$ matrix, $B$ be a $n \times p$ matrix, and $b_1, \ldots, b_p$ are columns of $B$, then the matrix $AB = [Ab_1, \ldots, Ab_p]$ is a matrix of size $m \times p$.

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Let us compute $AB$ where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$, $Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$,

$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$. 
THEOREM

If the product $AB$ is defined, then the entry in row $i$ and column $j$ of $AB$ is the sum of the products of corresponding entries from row $i$ of $A$ and column $j$ of $B$. 
Row-Column Rule

THEOREM

If the product $AB$ is defined, then the entry in row $i$ and column $j$ of $AB$ is the sum of the products of corresponding entries from row $i$ of $A$ and column $j$ of $B$.

In other words if $(AB)_{i,j}$ denotes $(i,j)$-entry of $AB$ and $A$ is $m \times n$ matrix, then

$$a_{i,1}b_{1,j} + \cdots + a_{i,n}b_{n,j}$$
THEOREM

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$$a_{i,1}b_{1,j} + \cdots + a_{i,n}b_{n,j}$$

THEOREM

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + BC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_mA = A = AI_n$
Powers of Matrix

DEFINITION

If $A$ is $n \times n$ matrix, then $A^k$ denotes the product of $k$ copies of $A$. 
The Transpose of Matrix

DEFINITION

If $A$ is $m \times n$ matrix, then the transpose of $A$, denoted by $A^T$ is a matrix whose columns are formed from the corresponding rows of $A$. 
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**EXAMPLE**

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Then \( A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \).
The Transpose of Matrix

**DEFINITION**

If $A$ is $m \times n$ matrix, then the **transpose** of $A$, denoted by $A^T$ is a matrix whose columns are formed from the corresponding rows of $A$.

**THEOREM**

Let $A$ and $B$ denote matrices whose size are appropriate for the following operations.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. For any scalar $r$, $(rA)^T = r(A^T)$