Determinants and Volume

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THEOREM

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Columns Operations

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If A is a square matrix, then det A = det A^T.

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We prove this theorem using induction by size of A. Let A be an n × n matrix. For n = 1 the statement is obvious. Let us assume that we prove the statement for all smaller matrices and n = k + 1. Then cofactor of a_{1,j} in A is equal to cofactor of a_{j,1} in A^T.
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If $A$ is a square matrix, then $\det A = \det A^T$.

PROOF.

We prove this theorem using induction by size of $A$. Let $A$ be an $n \times n$ matrix. For $n = 1$ the statement is obvious. Let us assume that we prove the statement for all smaller matrices and $n = k + 1$. Then cofactor of $a_{1,j}$ in $A$ is equal to cofactor of $a_{j,1}$ in $A^T$. Hence the cofactor expansion of the first row of $A$ is equal to the cofactor expansion of the first column of $A^T$. \qed
The Multiplicative Property

THEOREM

Let $A$ and $B$ be $n \times n$ matrices. Then $\det AB = \det A \det B$.

EXAMPLE

Let $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.
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**EXAMPLE**

Let $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Then $AB = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$. 
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$\det AB = 25 \cdot 13 - 20 \cdot 14 = 45$
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$\det AB = 25 \cdot 13 - 20 \cdot 14 = 45$ and $(\det A) \cdot (\det B) = 9 \cdot 5 = 45$. 
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- If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det B = \det A$.
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- If two rows of $A$ interchanged to produce $B$, then $\det B = -\det A$. 
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- If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \det A$. 

The Multiplicative Property

**THEOREM**

If $A$ is an $n \times n$ matrix and $E$ is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 
1 & \text{if } E \text{ is a row replacement} \\
-1 & \text{if } E \text{ is an interchange} \\
r & \text{if } E \text{ is a scale by } r 
\end{cases}$$
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We prove this theorem using induction by $n$. It is easy to check this statement for $n = 2$. Let us prove the induction step. Let the statement holds for all matrices of size $k \times k$ where $k = n - 1$. 
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We prove this theorem using induction by \( n \). It is easy to check this statement for \( n = 2 \).

Let us prove the induction step. Let the statement holds for all matrices of size \( k \times k \) where \( k = n - 1 \).

Let \( i \) be a row that was not changed by the row operation \( E \).
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We prove this theorem using induction by $n$. It is easy to check this statement for $n = 2$.
Let us prove the induction step. Let the statement holds for all matrices of size $k \times k$ where $k = n - 1$.
Let $i$ be a row that was not changed by the row operation $E$. Let $A_{i,j}$ ($B_{i,j}$ respectively) be the matrix obtained by deleting $i$th row and $j$th column from $A$ ($EA$ respectively).
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Note that rows of $B_{i,j}$ are obtained from $A_{i,j}$ by the same type of row operation that $E$ performs on $A$. 
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Using cofactor expansion
$\det EA = a_{i,1} \det B_{i,1} - a_{i,2} \det B_{i,2} + \cdots + (-1)^n a_{i,n} \det B_{i,n}$
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Note that rows of \( B_{i,j} \) are obtained from \( A_{i,j} \) by the same type of row operation that \( E \) performs on \( A \). Hence \( \det B_{i,j} = \alpha \det A_{i,j} \) where \( \alpha = \det E \).
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\det EA = a_{i,1} \det B_{i,1} - a_{i,2} \det B_{i,2} + \cdots + (-1)^n a_{i,n} \det B_{i,n} = \
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Let $A$ and $B$ be $n \times n$ matrices. Then $\det AB = \det A \det B$.

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Note that if $A$ is not invertible, then $AB$ is not invertible too.
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Cramer’s Rule

For any $n \times n$ matrix $A$ and $b \in \mathbb{R}^n$, let $A_i(b)$ be a matrix obtained form $A$ by replacing column $i$ of $A$ by $b$. 
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**THEOREM**

Let $A$ be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution $x$ of $Ax = b$ has entries defined by $x_i = \frac{\det A_i(b)}{\det A}$.
EXAMPLE

Let us solve the system

\[3sx - 2y = 4\]
\[-6x + sy = 1\]
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View the system as \(Ax = b\) where \(A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}\)
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Let us solve the system

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View the system as \( Ax = b \) where

\[
A = \begin{bmatrix}
3s & -2 \\
-6 & s
\end{bmatrix}
\]

and

\[
A_1(b) = \begin{bmatrix}
4 & -2 \\
1 & s
\end{bmatrix}, \quad A_2(b) = \begin{bmatrix}
3s & 4 \\
-6 & 1
\end{bmatrix}.
\]
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Since \(\det A = 3s^2 - 12 = 3(s + 2)(s - 2)\) the system has unique solution when \(s \neq \pm 2\).
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For such \( s \) the solution is

\[ \begin{align*}
x_1 &= \frac{\det A_1(b)}{\det A} = \frac{4s + 2}{3(s + 2)(s - 2)}, \\
x_2 &= \frac{\det A_2(b)}{\det A} = \frac{3s + 24}{3(s + 2)(s - 2)} = \frac{s + 8}{(s + 2)(s - 2)}.
\]