1. (20 points) Prove the following equalities.

(a) \[ 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \]

**Solution:** We proceed by induction on \( n \). The base case \( (n = 1) \) holds since
\[
1^2 = \frac{1 \cdot (1 + \frac{1}{2}) \cdot 2}{3}.
\]
Now, for \( n \geq 1 \), we suppose that \( 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n \) and seek to show the same identity for the \((n+1)\)st case. We have that
\[
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \left[ \frac{n(n+1)(2n+1)}{6} \right] + (n+1)^2
\]
\[
= (n+1) \cdot \left( \frac{n(n+1/2)(n+1)}{3} + (n+1) \right)
\]
\[
= (n+1) \cdot \left( \frac{n(n+1/2) + 3n + 3}{3} \right)
\]
\[
= (n+1) \cdot \left( \frac{n^2 + \frac{n}{2} + 3n + 3}{3} \right)
\]
\[
= (n+1) \cdot \left( \frac{(n+3/2)(n+2)}{3} \right) = \frac{(n+1)(n+3/2)(n+2)}{3}
\]

(b) \[ 1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2. \]

**Solution:** We proceed by induction on \( n \). The base case \( (n = 1) \) holds since
\[
1^3 = \left( \frac{1 \cdot 2}{2} \right)^2.
\]
Now, for \( n \geq 1 \), we suppose that \( 1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2 \) for \( n \) and seek to show the same identity for the \((n+1)\)st case. We have that
\[
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = \left( \frac{n(n+1)}{2} \right)^2 + (n+1)^3
\]
\[
= (n+1)^2 \left( \frac{n^2}{4} + n + 1 \right)
\]
\[
= (n+1)^2 \left( \frac{n^2 + 4n + 4}{4} \right) = \left( \frac{(n+1)(n+2)}{2} \right)^2
\]
2. (20 points) Prove that for every integers $a_1, \ldots, a_n$ there are $k > 0$ and $\ell \geq 0$ such that $k + \ell \leq n$ and $\sum_{i=0}^{\ell} a_{k+i}$ is divisible by $n$.

Solution: We consider the $n$ sums modulo $n$ of the form $(a_1), (a_1 + a_2), \ldots, (a_1 + a_2 \cdots + a_n)$. First, we note that if any of these sums $a_1 + \cdots + a_\ell \equiv 0 \pmod{n}$, then we are done by picking $k = 1$ and $\ell$ accordingly since being equivalent to 0 (mod $n$) is the same as being divisible by $n$. As a result, it suffices to show the result when none of the $n$ sums of the form $(a_1), (a_1 + a_2), \ldots, (a_1 + a_2 \cdots + a_n)$ are equivalent to 0 (mod $n$).

In this case, each of these sums modulo $n$ necessarily must be one of 1, 2, \ldots, $(n-1)$. That is, there are $(n-1)$ possible values for each of these $n$ sums. Therefore, by the pigeonhole principle, we must have that two of these sums are equivalent modulo $n$. We thus have that there exists $m > 0$ and $j > 0$ (and without loss of generality may assume that $j > m$) so that

$$a_1 + a_2 + \cdots + a_m \equiv a_1 + a_2 + \cdots + a_j \pmod{n}.$$

Now, we note that subtracting $a_1 + a_2 + \cdots + a_m$ from both sides yields that

$$0 \equiv a_{m+1} + \cdots + a_j \pmod{n}$$

and hence by taking $k = m + 1$ and $\ell = j - m - 1 = j - k$ we have that

$$0 \equiv a_k + a_{k+1} + \cdots + a_{k+\ell} \pmod{n}$$

which gives is that $\sum_{i=0}^{\ell} a_{k+i}$ is divisible by $n$ as desired.
3. (10 points) How many 6-digit numbers are there that have the same remainder modulo 2 of all the digits?

**Solution:** We will give two solutions depending on one’s interpretation of 6-digit numbers (i.e. whether or not the leading coefficient can be 0).

First, in the case where the leading coefficient may be 0, there are 10 choices for the first digit. Upon picking the first digit, we now must pick digits that have the same parity (i.e. even or odd) as the first digit. Thus, there are 5 choices for digits two through six, and we get

\[10 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 31250 \text{ total digits.}\]

However, if you interpreted the problem as the leading coefficient cannot be 0, then there would be 9 options for the first digit and then 5 for each of the remaining. We would then get

\[9 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 28125 \text{ total digits.}\]
4. (20 points) How many pairs of subsets $A, B \subseteq [n]$ are there such that $A \cap B \neq \emptyset$.

**Solution:** We let $\text{NEIP}(n)$ be the number of pairs $(A, B)$ where $A, B \subseteq [n]$ are such that $A \cap B \neq \emptyset$. (where NEIP stands for Non-Empty Intersection Property.) Similarly, we let $\text{EIP}(n)$ be the number of pairs $(A, B)$ of subsets such that $A \cap B = \emptyset$. We now note that given a pair of subsets $(A, B)$ we either have that $A \cap B = \emptyset$ or $A \cap B \neq \emptyset$ and cannot possibly have both. As a result,

$$\text{NEIP}(n) + \text{EIP}(n) = 4^n$$

where $4^n$ is the total number of pairs $(A, B)$ of subsets since there are $2^n$ choices for each coordinate and $2^n \cdot 2^n = 4^n$.

As a result, in order to find $\text{NEIP}(n)$, it suffices to count $\text{EIP}(n)$. Let us fix some $(A, B)$ such that $A \cap B = \emptyset$ and note that for each element of $x \in [n]$ there are three possibility

1. $x \in A$ and $x \notin B$,
2. $x \notin A$ and $x \in B$,
3. $x \notin A$ and $x \notin B$.

As a result, there are $3^n$ variants of $(A, B)$ such that $A \cap B = \emptyset$; i.e. $\text{EIP}(n) = 3^n$.

Hence, we get that

$$\text{NEIP}(n) = 4^n - \text{EIP}(n) = 4^n - 3^n.$$