1. We call a partition \( \{P_1, \ldots, P_k\} \) of \([n]\) nice iff \((j + 1) \not\in P_i\) for every \(i \in [k]\) and \(j \in P_i\). Prove that number of nice partitions is equal to \(B(n - 1)\).

**Solution:** Define \(S_n = \) the set of all partitions of \([n]\) and \(M_n = \) the set of all nice partitions of \([n]\). We are going to construct a bijection

\[
f : S_{n-1} \rightarrow M_n.
\]

Notice that we can obtain every nice partition of \([n]\) by:

1. First adding a singleton block \(\{n\}\) to any partition of \([n]\).
2. Then, in each block of that partition of \([n - 1]\), we locate consecutive integers \(i, i + 1, \ldots, i + j\) and if \(j\) is odd place every other integer, i.e. \(i, i + 2, i + 4, \ldots, i + j - 1\), into the block with \(n\), if \(j\) is even we put \(i + 1, \ldots, i + j - 1\) into the block with \(n\). We do this for each consecutive sequence of each block of each partition of \([n - 1]\) to obtain all possible nice partitions of \([n]\).

Note that the resulting partition is nice, since \(i + j \leq n - 1\), hence \(i + j - 1 < n - 1\).

For each nice partition of \([n]\) obtained this way we can have an inverse transformation \(f^{-1}\) that takes every nice partition of \([n]\) and gives the corresponding original partition of \([n - 1]\) by:

1. Taking every element except \(n\) in the block that contains \(n\) and placing each element \(i\), in order, into a block that contains \(i - 1\).
2. We then remove the block of \(n\) from our partition resulting in a partition of \([n - 1]\).

It can be seen that applying \(f^{-1}\) to every partition of \(M_n\) gives us every partition of \(S_n\). Thus \(f\) forms a bijection between \(M_n\) and \(S_{n-1}\).
2. How many different 6-digit numbers have sum of their digits at most 47?

**Solution:** Let $F$ denote the number of 6-digit numbers whose sum is less than 48. Note that number $G$ of 6-digit numbers with sum of digits at least 48 is equal to $9 \cdot 10^5 - F$.

Let us find $G$. Note that the maximal sum of digits of a 6-digit number is $6 \cdot 9 = 54$. Hence, in order to transform 999999 into another number with sum of digits at least 48 we need to subtract $54 - 48 = 6$ from the digits of 999999; i.e. we need to put at most $54 - 48 = 6$ balls into 6 boxes (each digit corresponds to a box).

Hence, $G = \sum_{i=0}^{6} \binom{6+i-1}{i} = \sum_{i=0}^{6} \binom{5+i}{5} = \binom{12}{6} = 924$. Thus $F = 9 \cdot 10^5 - 924 = 899076$. 
3. How many ways to put $n$ indistinguishable balls into $k$ different boxes if we have to put at least $a_i$ balls into the box with number $i$.

\begin{center}
\textbf{Solution:} Let $i \in [k]$ and $a_i$ denote the minimum number of balls the $i$th box contains for all $a_1, \ldots, a_k$. So we can place $a_i$ balls into the $i$th box for all $i$. Let $j = a_1 + a_2 + \cdots + a_k$ then we have $n - j$ balls left to place into $k$ boxes. This is then a weak composition problem, thus the solution is $\binom{n-j+k-1}{k-1}$.
\end{center}