1. (10 points) Find a closed formula (no summation signs) for the expression $c(n, n - 3)$.

**Solution:** Let us consider the following cases:

1. The longest cycle has length at least 5, note that it is impossible, since after deleting this cycle you have $n - 5$ elements distributed by $n - 4$ cycles.

2. The longest cycle has length 4, in this case all other cycles have length 1. Hence, there are $\binom{n}{4} \cdot 3!$ permutations like this.

3. The longest cycle has length 3, in this case there is a cycle of length 2. Hence, there are $\binom{n}{3} \cdot \binom{n-3}{2} \cdot 2!$ permutations like this.

4. The longest cycle has length 2, in this case we have 3 cycles of length 2 and all other cycles have length 1. Hence, there are $\binom{n}{2} \cdot \binom{n-2}{2} \cdot \binom{n-4}{2}/3!$.

As a result the final answer is $\binom{n}{4} \cdot 3! + \binom{n}{3} \cdot \binom{n-3}{2} \cdot 2! + \binom{n}{3} \cdot \binom{n-3}{2} \cdot \binom{n-4}{2}/3!$. 
2. (10 points) Once upon a time Skeeve made a mistake and borrowed 1000 golden coins from Aahz. As you probably know, Aahz is very greedy, so he set the interest rate to be equal 10% per day. Every morning Skeeve pay 99 golden coins. Write a closed formula for Skeeve debt.

**Solution:** Note that amount of money Skeeve owe on \(n\)th day is satisfying \(a_0 = 1000\) and \(a_n = 1.1a_{n-1} - 99\). Hence, \(a_n x^n = 1.1a_{n-1} x^n - 99x^n\). Let us define \(f(x) = \sum_{n=0}^{\infty} a_n x^n\). We may note that

\[
\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 1.1a_{n-1} x^n - 99x^n.
\]

Thus \(f(x) - a_0 = 1.1x f(x) - \frac{99}{1-x}\). Using this equality we may derive that

\[
f(x) = \frac{a_0}{1-x} - \frac{99}{(1-x)(1.1x)} = \frac{a_0}{1-x} - \frac{990}{1-x} + \frac{990}{1-1.1x} = \frac{10}{1-x} + \frac{990}{1-1.1x} = \sum_{n=0}^{\infty} 10x^n + 990 \cdot (1.1)^n x^n.
\]

Hence, \(a_n = 10 + 990 \cdot (1.1)^n\).
3. (10 points) In Durmstrang Institute every semester consists of two parts with two midterms in each of them. Find a closed formula for number of ways to organize semester if there are \( n \) days in this semester.

**Solution:** Note that in the class we proved that 
\[
\sum_{n=0}^{\infty} \binom{n}{2} x^n = \frac{x^2}{(1-x)^3}.
\]
Additionally, note that number of ways \( a_n \) to organize a semester consisting of \( n \) days is equal to \( \sum_{k=0}^{n} \binom{k}{2} \cdot \binom{n-k}{2} \).

Hence, if we define 
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n,
\]
then 
\[
f(x) = \left( \sum_{n=0}^{\infty} \binom{n}{2} x^n \right)^2; \text{ i.e. } f(x) = \frac{x^4}{(1-x)^7}.
\]
Thus, 
\[
f(x) = \sum_{n=0}^{\infty} \binom{n+1}{5} x^n.
\]
As a result, 
\[a_n = \binom{n+1}{5}.\]