1. (10 points) Find a closed formula (no summation signs) for the expression $c(n, n - 3)$.

**Solution:** Let us consider the following cases:

1. The longest cycle has length at least 5, note that it is impossible, since after deleting this cycle you have $n - 5$ elements distributed by $n - 4$ cycles.

2. The longest cycle has length 4, in this case all other cycles have length 1. Hence, there are \( \binom{n}{4} \cdot 3! \) permutations like this.

3. The longest cycle has length 3, in this case there is a cycle of length 2. Hence, there are \( \binom{n}{3} \cdot \binom{n-3}{2} \cdot 2! \) permutations like this.

4. The longest cycle has length 2, in this case we have 3 cycles of length 2 and all other cycles have length 1. Hence, there are \( \binom{n}{2} \cdot \binom{n-2}{2} \cdot \binom{n-4}{2} / 3! \) permutations like this.

As a result the final answer is \( \binom{n}{4} \cdot 3! + \binom{n}{3} \cdot \binom{n-3}{2} \cdot 2! + \binom{n}{3} \cdot \binom{n-2}{2} \cdot \binom{n-4}{2} / 3! \).
2. (10 points) Once upon a time Skeeve made a mistake and borrowed 1000 golden coins from Aahz. As you probably know, Aahz is very greedy, so he set the interest rate to be equal 10% per day. Every morning Skeeve pay 99 golden coins. Write a closed formula for Skeeve debt.

**Solution:** Note that amount of money Skeeve owe on \( n \)th day is satisfying \( a_0 = 1000 \) and \( a_n = 1.1a_{n-1} - 99 \). Hence, \( a_nx^n = 1.1a_{n-1}x^n - 99x^n \). Let us define \( f(x) = \sum_{n=0}^{\infty} a_n x^n \). We may note that \( \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 1.1a_{n-1}x^n - 99x^n \). Thus \( f(x) - a_0 = 1.1xf(x) - \frac{99}{1-x} \). Using this equality we may derive that

\[
\begin{align*}
f(x) &= \frac{a_0}{1-x} - \frac{99}{(1-x)(1-1.1x)} = \frac{a_0}{1-x} - \frac{990}{1-x} + \frac{990}{1-1.1x} = \\
&\quad \frac{10}{1-x} + \frac{990}{1-1.1x} = \sum_{n=0}^{\infty} 10x^n + 990 \cdot (1.1)^n x^n.
\end{align*}
\]

Hence, \( a_n = 10 + 990 \cdot (1.1)^n \).
3. (10 points) In Durmstrang Institute every semester consists of two parts with two midterms in each of them. Find a closed formula for number of ways to organize semester if there are \( n \) days in this semester.

Solution: Note that in the class we proved that

\[
\sum_{n=0}^{\infty} \binom{n}{2} x^n = \frac{x^2}{(1-x)^3}.
\]

Additionally, note that number of ways \( a_n \) to organize a semester consisting of \( n \) days is equal to

\[
\sum_{k=0}^{n} \binom{k}{2} \cdot \binom{n-k}{2}.
\]

Hence, if we define \( f(x) = \sum_{n=0}^{\infty} a_n x^n \), then

\[
f(x) = \left( \sum_{n=0}^{\infty} \binom{n}{2} x^n \right)^2 \text{; i.e. } f(x) = \frac{x^4}{(1-x)^3}.
\]

Thus, \( f(x) = \sum_{n=0}^{\infty} \binom{n+1}{5} x^n \).

As a result, \( a_n = \binom{n+1}{5} \).