1. (10 points) We say that a Boolean function \( f(x_1, \ldots, x_n) \) \((f : \{0, 1\}^n \rightarrow \{0, 1\})\) depends on \( x_i \) iff there are \( v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \) such that
\[
f(v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_n) \neq f(v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_n).
\]
Find a closed formula (with one summation sign from 0 to \( n \)) for number of functions that depends on all their inputs.

**Solution:** For \( n \) Boolean variables \( x_1, \ldots, x_n \) there are \( 2^n \) possible different inputs. In order to obtain all possible Boolean functions of \( n \) we notice that each input can contribute either 0 or 1 to the function. So the total number of all Boolean functions \( f(x_1, \ldots, x_n) \) is equal to \( 2^{2^n} \).

Applying the inclusion-exclusion principle we first find all the functions that are not valid for each \( x_i \). The number of Boolean functions that do not depend on some \( x_i \) is \( 2^{2^{n-1}} \) as we choose values for each of the remaining \( n-1 \) variables. Thus we have \( \binom{n}{1}2^{2^{n-1}} \). For the number of Boolean functions that don’t depend on some \( \{x_{i_1}, \ldots, x_{i_k}\} \) using similar arguments we have \( \binom{n}{k}2^{2^{n-k}} \). Thus the number of functions \( f(x_1, \ldots, x_n) \) that depend on all \( n \) variables \( x_1, \ldots, x_n \) using the inclusion-exclusion principle:
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{2^{n-k}}
\]
2. (10 points) How many numbers from 1 to 1000 are neither square numbers nor cubic numbers?

**Solution:** Let \( A_i = \{ x : x \in [1000] \text{ and } \sqrt[i]{x} \in \mathbb{Z} \} \). Want to find \( 1000 - |A_2 \cup A_3| \). Note that \( |A_i| = \sqrt[1000]{1000} \). Hence,

\[
|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = |A_2| + |A_3| - |A_6| = \left\lfloor \sqrt[1000]{1000} \right\rfloor + \left\lfloor \sqrt[1000]{1000} \right\rfloor - \left\lfloor \sqrt[1000]{1000} \right\rfloor = 31 + 10 - 3 = 38.
\]

Hence, the answer is equal to \( 1000 - 38 = 962 \).
3. (10 points) Let \( r_1 \) and \( r_2 \) are solutions of the equation \( \lambda^2 - b_1 \lambda - b_2 = 0 \) and \( r_1 \neq r_2 \) i.e. \( b_1 = r_1 + r_2 \) and \( b_2 = -r_1 r_2 \). Find a closed formula (no summation signs) for the recurrent sequence \( a_n \) such that \( a_{n+2} = b_1 a_{n+1} + b_2 a_n \) for \( n \geq 0 \), \( a_1 = r_1 + r_2 \), and \( a_0 = 2 \).

**Solution:** Let us consider \( A(x) = \sum_{n=0}^{\infty} a_n x^n \). Using recurrence relation we get the following equality:

\[
A(x) = 2 + (r_1 + r_2)x + \sum_{n=2}^{\infty} (b_1 a_{n-1} + b_2 a_{n-2})x^n =
2 + b_1 x + b_1 \left( \sum_{n=0}^{\infty} a_n x^{n+1} - 2x \right) + b_2 \left( \sum_{n=0}^{\infty} a_n x^{n+2} \right) =
2 + b_1 x + b_1 (xA(x) - 2x) + b_2 x^2 A(x)
\]

Hence, \( A(x) - b_1 x A(x) - b_2 x^2 A(x) = 2 + b_1 x - 2b_1 x \); and

\[
A(x) = \frac{2 - b_1 x}{1 - b_1 x - b_2 x^2} = \frac{2 - (r_1 + r_2)x}{1 - (r_1 + r_2)x + r_1 r_2 x^2} = \frac{2 - (r_1 + r_2)x}{(r_1 x - 1)(r_2 x - 1)} = \frac{1}{1-r_1 x} + \frac{1}{1-r_2 x}.
\]

Since \( \frac{1}{1-r_1 x} + \frac{1}{1-r_2 x} = \sum_{n=0}^{\infty} r_1^n x^n + \sum_{n=0}^{\infty} r_2^n x^n \), \( A(x) = \sum_{n=0}^{\infty} r_1^n x^n + r_2^n x^n \) and the answer is \( a_n = r_1^n + r_2^n \).