1. Let $G$ be a graph on $n$ vertices such that every vertex has odd degree. Show that $n$ is even.

**Solution:** Notice for any odd and even numbers the following identities hold: odd + odd = even and even + odd = odd.

Assume there are $k$ edges in $G$ where $k \in \mathbb{Z}$ and let $v_i$ denotes the $i^{th}$ vertex of $G$ and $d_i$ denotes degree of it where $i \in [n]$. Then the total degree of $G$ (i.e. $\sum_{i=1}^{n} d_i$) is $2k$ since we count every edge twice. Since every vertex has odd degree $2k = \sum_{i=1}^{n} d_i = \text{odd} + \text{odd} + \cdots + \text{odd}$.

Notice if $n$ is odd then there are an odd number of vertices and $\sum_{i=1}^{n} d(v_i) = \text{odd} \neq 2k$ as $2k$ is even. Thus $n$ must be even.
2. Find a minimal $k(n)$ such that for every graph $G$ on $n$ vertices, if $G$ has at least $k(n)$ edges, then $G$ is connected.

**Solution:** Assume $G$ is a simple graph. The total possible edges that $G$ can have is $\binom{n}{2}$ as each edge represents a connection between two vertices and there are $\binom{n}{2}$ sets of two vertices.

Let us assume that $G$ is not connected i.e. we may split $G$ into two disjoint subgraphs $G_1$ and $G_2$ of $G$ such that vertices of $G_1$ are not connected with vertices of $G_2$. Let $n_1$ and $n_2$ denote the number of vertices in $G_1$ and $G_2$ respectively. Note that $G_i$ has at most $\binom{n_i}{2}$ edges for every $i \in [2]$. Hence, $G$ has at most $\binom{n_1}{2} + \binom{n_2}{2}$ edges.

However, $f(n_1) = \binom{n_1}{2} + \binom{n-n_1}{2}$ reached the minimum for $1 \leq n_1 \leq n - 1$ for $n_1 = 1$. As a result, $k(n) \leq \binom{n-1}{2} + 1$.

Additionally, note that there is a not connected graph $G$ with $\binom{n-1}{2}$ edges (it is a complete graph on $n - 1$ vertices and one additional vertex). Thus $k(n) = \binom{n-1}{2} + 1$. 
3. Let $G$ be a graph. Show that there are two different vertices $u$ and $v$ such that they have the same degree.

**Solution:** Assume $G$ is a simple graph with $n$ vertices. Suppose every vertex in $G$ has different degrees. The set of degrees of the vertices is then $\{0, 1, \ldots, n-1\}$. However this means that one vertex is connected to all other vertices and another vertex is connected to no other vertices, it is a contradiction. Therefore there are at least two different vertices $u$ and $v$ such that they have the same degree.