1. (50 points) Check all the correct statements.

■ The number of different strings you can get by reordering letters in the word aabbc is 30.
□ There are 25 different strings of length 5 over the alphabet with two letters.
■ If you have 26 balls in 5 boxes, then there is a box with at least 6 balls.
■ There are 6 different surjective functions from [3] to [2].
■ There are 15 variants to put 4 identical balls into 3 different boxes.

2. (10 points) Let us assume that we are given $\ell$ lines that are not parallel to each other. Prove that there are at least two of them such that angle between them is at most $\pi/\ell$.

Solution: Move all the lines (using parallel shift) such that all of them are going through $(0,0)$. Let us denote angles between lines (in clockwise order) $\alpha_1, \ldots, \alpha_{2\ell}$ respectively and assume that all of them are greater than $\pi/\ell$. In this case we may note that $\sum_{i=1}^{2\ell} \alpha_i > 2\ell \cdot \pi/\ell = 2\pi$, but we know that $\sum_{i=1}^{2\ell} \alpha_i = 2\pi$. 
3. (10 points) Prove that for all integers $n > 0$, the sum $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ is at most 2.

**Solution:** Let us prove a stronger statement:

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

We prove this statement by induction, basis is clear. Now we need to prove induction step. By induction hypothesis

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n},$$

Note that $\frac{1}{n} - \frac{1}{(n+1)^2} \geq \frac{1}{n+1}$ (since $(n+1)^2 - n \geq n(n+1)$). As a result,

$$\sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$
4. (10 points) Find a closed formula (no summation signs) for the expression \(\sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i\).

**Solution:** Firstly, let us consider cases when \(n \leq 2\). If \(n = 0\), then \(\sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i = 0\). If \(n = 1\), then \(\sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i = -1\). If \(n = 2\), then \(\sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i = 2\).

Let us now consider other cases. Note that

\[(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.\]

Hence, if we derive both sides of the equality we get

\[n \cdot (1 + x)^{n-1} = \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1}.\]

Hence, if we substitute \(x = -1\) we prove that \(\sum_{i=1}^{n} i \cdot \binom{n}{i} (-1)^{i-1} = 0\). Now let us derive the equality once again:

\[n(n-1) \cdot (1 + x)^{n-2} = \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} x^{i-2}.\]

Using previous argument we prove that \(\sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} (-1)^i = 0\). Since \(\sum_{i=0}^{n} \binom{n}{i} x^i = x^2 \cdot \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} x^{i-2} + x \cdot \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1}\), the answer is 0.
5. (10 points) Find a closed formula (no summation signs) for the expression $S(n, n - 2)$.

**Solution:** Note that there are two variants how we can split $n$ elements into $n - 2$ subsets:

1. all subsets except one are singletons, there are $\binom{n}{3}$ ways to do this;
2. all subsets except two are singletons, there are $\binom{n}{4} \cdot \binom{4}{2} \frac{1}{2}$ (we divide by two since the order of these two sets of size 2 is not important).

Hence, the answer is $\binom{n}{3} + \binom{n}{4} \cdot \binom{4}{2} \frac{1}{2}$. 