1. (50 points) Check all the correct statements.

■ The number of different strings you can get by reordering letters in the word aabbc is 30.
□ There are 25 different strings of length 5 over the alphabet with two letters.
■ If you have 26 balls in 5 boxes, then there is a box with at least 6 balls.
■ There are 6 different surjective functions from \([3]\) to \([2]\).
■ There are 15 variants to put 4 identical balls into 3 different boxes.

2. (10 points) Let us assume that we are given \(\ell\) lines that are not parallel to each other. Prove that there are at least two of them such that angle between them is at most \(\pi/\ell\).

Solution: Move all the lines (using parallel shift) such that all of them are going through \((0, 0)\). Let us denote angles between lines (in clockwise order) \(\alpha_1, \ldots, \alpha_{2\ell}\) respectively and assume that all of them are greater than \(\pi/\ell\). In this case we may note that \(\sum_{i=1}^{2\ell} \alpha_i > 2\ell \cdot \pi/\ell = 2\pi\), but we know that \(\sum_{i=1}^{2\ell} \alpha_i = 2\pi\).
3. (10 points) Prove that for all integers $n > 0$, the sum $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ is at most 2.

**Solution:** Let us prove a stronger statement:

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$  

We prove this statement by induction, basis is clear. Now we need to prove induction step. By induction hypothesis

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n},$$  

Note that $\frac{1}{n} - \frac{1}{(n+1)^2} \geq \frac{1}{n+1}$ (since $(n+1)^2 - n \geq n(n+1)$). As a result,

$$\sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$  

4. (10 points) Find a closed formula (no summation signs) for the expression \( \sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i \).

**Solution:** Let us note that

\[
(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.
\]

Hence, if we derive both sides of the equality we get

\[
n \cdot (1 + x)^{n-1} = \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1}.
\]

Hence, if we substitute \( x = -1 \) we prove that \( \sum_{i=1}^{n} i \cdot \binom{n}{i} (-1)^{i-1} = 0 \). Now let us derive the equality once again:

\[
n(n-1) \cdot (1 + x)^{n-2} = \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} x^{i-2}.
\]

Using previous argument we prove that \( \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} (-1)^i = 0 \). Since \( \sum_{i=0}^{n} \binom{n}{i} x^i = x^2 \cdot \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} x^{i-2} + x \cdot \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1} \), the answer is 0.
5. (10 points) Find a closed formula (no summation signs) for the expression $S(n, n - 2)$.

**Solution:** Note that there are two variants how we can split $n$ elements into $n - 2$ subsets:

1. all subsets except one are singletons, there are $\binom{n}{3}$ ways to do this;
2. all subsets except two are singletons, there are $\binom{n}{4} \cdot \binom{4}{2}$ (we divide by two since the order of these two sets of size 2 is not important).

Hence, the answer is $\binom{n}{3} + \binom{n}{4} \cdot \binom{4}{2}$. 