Name: 

Pid: 

1. (50 points) Check all the correct statements.
- ■ The number of different strings you can get by reordering letters in the word aabbc is 30.
- □ There are 25 different strings of length 5 over the alphabet with two letters.
- ■ If you have 26 balls in 5 boxes, then there is a box with at least 6 balls.
- ■ There are 6 different surjective functions from [3] to [2].
- ■ There are 15 variants to put 4 identical balls into 3 different boxes.

2. (10 points) Let us assume that we are given ℓ lines that are not parallel to each other. Prove that there are at least two of them such that angle between them is at most π/ℓ.

Solution: Move all the lines (using parallel shift) such that all of them are going through (0, 0). Let us denote angles between lines (in clockwise order) α₁, . . . , α₂ℓ respectively and assume that all of them are greater than π/ℓ. In this case we may note that \( \sum_{i=1}^{2\ell} \alpha_i > 2\ell \cdot \pi/\ell = 2\pi \), but we know that \( \sum_{i=1}^{2\ell} \alpha_i = 2\pi \).
3. (10 points) Prove that for all integers $n > 0$, the sum $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ is at most 2.

**Solution:** Let us prove a stronger statement:

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$  

We prove this statement by induction, basis is clear. Now we need to prove induction step. By induction hypotheses

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$  

Note that $\frac{1}{n} - \frac{1}{(n+1)^2} \geq \frac{1}{n+1}$ (since $(n+1)^2 - n \geq n(n+1)$). As a result,

$$\sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$  

4. (10 points) Find a closed formula (no summation signs) for the expression \( \sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i \).

Solution: Firstly, let us consider cases when \( n \leq 2 \). If \( n = 0 \), then \( \sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i = 0 \). If \( n = 1 \), then \( \sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i = -1 \). If \( n = 2 \), then \( \sum_{i=1}^{n} i^2 \binom{n}{i} (-1)^i = 2 \).

Let us now consider other cases. Note that

\[
(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.
\]

Hence, if we derive both sides of the equality we get

\[
n \cdot (1 + x)^{n-1} = \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1}.
\]

Hence, if we substitute \( x = -1 \) we prove that \( \sum_{i=1}^{n} i \cdot \binom{n}{i} (-1)^{i-1} = 0 \). Now let us derive the equality once again:

\[
n(n - 1) \cdot (1 + x)^{n-2} = \sum_{i=2}^{n} i(i - 1) \cdot \binom{n}{i} x^{i-2}.
\]

Using previous argument we prove that \( \sum_{i=2}^{n} i(i - 1) \cdot \binom{n}{i} (-1)^i = 0 \). Since \( \sum_{i=0}^{n} \binom{n}{i} x^i = x^2 \cdot \sum_{i=2}^{n} i(i - 1) \cdot \binom{n}{i} x^{i-2} + x \cdot \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1} \), the answer is 0.
5. (10 points) Find a closed formula (no summation signs) for the expression $S(n, n - 2)$.

**Solution:** Note that there are two variants how we can split $n$ elements into $n - 2$ subsets:

1. all subsets except one are singletons, there are $\binom{n}{3}$ ways to do this;
2. all subsets except two are singletons, there are $\binom{n}{4} \cdot \binom{4}{2} \frac{1}{2}$ (we divide by two since the order of these two sets of size 2 is not important).

Hence, the answer is $\binom{n}{3} + \binom{n}{4} \cdot \binom{4}{2} \frac{1}{2}$. 