1. (80 points) Check all the correct statements.
   ■ There are 6 permutations of [4] with 3 cycles.
   ■ The number of different strings you can get by reordering letters in the word aabbc is 30.
   ■ The following graph is connected.

   D
   /|
  / |
 /  |
A—B—C

   □ There are 125 different strings of length 5 over the alphabet with 3 letters.
   ■ If you have 26 balls in 5 boxes, then there is a box with at least 6 balls.
   ■ There are 6 different surjective functions from [3] to [3].
   □ The following graph has an Euler path.

   D
   /|
  / |
 /  |
A—B—C

   E

   ■ There are 15 variants to put 4 identical balls into 3 different boxes.
2. (10 points) Let \( a_n = 2a_{n-1} - a_{n-2} \) for \( n \geq 2 \), \( a_1 = 2 \), and \( a_0 = 1 \). Find a closed formula (no summation signs) for the recurrent sequence \( a_n \).

Solution: Let us define \( f(x) = \sum_{n=0}^{\infty} a_n x^n \). Note that \( a_n x^n = 2a_{n-1}x^n - a_{n-2}x^n \); hence, \( f(x) - a_1 x - a_0 = 2xf(x) - 2a_0 x - x^2f(x) \). As a result, \( f(x) - 2xf(x) + x^2f(x) = 1 \).

We may conclude that \( f(x) = \frac{1}{(1-x)^2} \). In class we proved that it implies that \( f(x) = \sum_{n=0}^{\infty} (n+1)x^n \).

Thus \( a_n = n + 1 \).
3. (10 points) Let $i_1, \ldots, i_k \in [n]$ be some different integers. Find a closed formula (no summation signs) for number of permutations $\pi$ such that 

$$\pi^{-1}(i_1) \leq \pi^{-1}(i_2) \leq \cdots \leq \pi^{-1}(i_k).$$

**Solution:** Let $1 \leq a_1 < a_2 \cdots < a_k \leq n$ and $A_{a_1, \ldots, a_k} = \{ \pi \in S_n \ : \ \forall j \in [k] \ \pi(a_j) = i_j \}$. It is easy to see that $|A_{a_1, \ldots, a_k}|$ is equal to the number of bijections from $[n] \setminus \{i_1, \ldots, i_k\}$ to $[n] \setminus \{i_1, \ldots, i_k\}$ i.e. $|A_{a_1, \ldots, a_k}| = (n-k)!$. Additionally, note that if $(a_1, \ldots, a_k) \neq (b_1, \ldots, b_k)$, then $A_{a_1, \ldots, a_k} \cap A_{b_1, \ldots, b_k} = \emptyset$.

As a result, 

$$\left| \bigcup_{1 \leq a_1 < a_2 \cdots < a_k \leq n} A_{a_1, \ldots, a_k} \right| = \binom{n}{k} (n-k)!. \text{ Hence, the answer is } \binom{n}{k} (n-k)! = \frac{n!}{k!}. $$