Section 4.2

1. (Problem 22b) Show \( f(x) = 4x^5 + 28x^4 + 7x^3 - 28x^2 + 14 \) is irreducible over \( \mathbb{Q}[x] \).

Solution: Notice that \( f(x) \in \mathbb{Z}[x] \). Furthermore notice 7 divides every coefficient except the leading term, and that \( 7^2 \) does not divide the constant term. Thus we can apply Eisenstein’s criterion with \( p = 7 \) to deduce that \( f(x) \) is irreducible in \( \mathbb{Q}[x] \).

2. (Problem 43c) If \( \sigma : \mathbb{F}[x] \to \mathbb{F}[x] \) is a ring automorphism that fixes \( \mathbb{F} \), show there exist \( a \in \mathbb{F} \setminus \{0\} \) and \( b \in \mathbb{F} \) such that \( \sigma(f) = f(ax + b) \) for all \( f \in \mathbb{F}[x] \).

Solution: The key point is that an automorphism of \( \mathbb{F}[x] \) which fixes \( \mathbb{F} \) is determined by the image of \( x \). To see what we mean, let \( f \in \mathbb{F}[x] \), say \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \). Using the fact that \( \sigma \) is a ring homomorphism, and that \( \sigma(a_i) = a_i \) for all \( i \) (because \( \sigma \) fixes \( \mathbb{F} \)), we calculate

\[
\sigma(f) = \sigma(a_0 + a_1 x + \cdots + a_n x^n) = \sigma(a_0) + \sigma(a_1)\sigma(x) + \cdots + \sigma(a_n)\sigma(x)^n = a_0 + a_1 \sigma(x) + \cdots + a_n \sigma(x)^n = f(\sigma(x)).
\]

Because this holds for any \( f \in \mathbb{F}[x] \), to complete the problem it suffices to show that there exist \( a, b \in \mathbb{F} \) such that \( \sigma(x) = ax + b \), or in other words it suffices to show that \( \deg(\sigma(x)) = 1 \). To see this, let \( p(x) = \sigma(x) \) and let \( q(x) = \sigma^{-1}(x) \). Then we calculate using the result of the calculation above (with \( f \) replaced by \( q \))

\[
x = \sigma(\sigma^{-1}(x)) = \sigma(q(x)) = q(\sigma(x)) = q(p(x)).
\]

Now we claim that \( \deg(q(p(x))) = \deg(q) \deg(p) \); this will imply that \( \deg(q) \deg(p) = 1 \), which lets us conclude \( \deg(q) = \deg(p) = 1 \), which completes the proof because we were supposed to show that \( \deg(\sigma(x)) = 1 \).

To prove the claim, let \( m = \deg(q) \) and \( n = \deg(p) \), so we can write \( p(x) = \sum_{i=0}^{n} a_i x^i \) and \( q(x) = \sum_{j=0}^{m} b_j x^j \) \( \text{where } a_n, b_m \neq 0 \). Then using the multinomial theorem we have

\[
q(p(x)) = \sum_{j=0}^{m} b_j (p(x))^j = \sum_{j=0}^{m} b_j (\sum_{i=0}^{n} a_i x^i)^j = \sum_{j=0}^{m} b_j \left( \sum_{k_0+\cdots+k_n=j} \binom{j}{k_0,\ldots,k_n} \prod_{i=0}^{n} (a_i x^i)^{k_i} \right).
\]

By inspection we see the largest power of \( x \) occurring in this expression is \( x^{nm} \), and the coefficient is \( a_1^{m_k} b_m \neq 0 \), which shows \( \deg(q(p(x))) = nm = \deg(q) \deg(p) \).
3. (Problem 1b) In each case find a monic polynomial \( h \) in \( F[x] \) such that \( I = \langle h \rangle \), where

\[
I = \{ f \in F[x] \mid \text{the sum of the coefficients of } f \text{ is zero} \}.
\]

**Solution:** We know there exists some monic polynomial \( h \) such that \( I = \langle h \rangle \). Notice that \( h \neq 0 \) because \( I \neq \{0\} \) (for instance \( x - 1 \in I \)), and also notice that \( h \) cannot be a constant because \( I \) does not contain any nonzero constant polynomials (this easily follows from the definition of \( I \)). Thus \( \deg(h) \geq 1 \). Now because \( x - 1 \in I = \langle h \rangle \), we can write \( x - 1 = h(x)q(x) \) for some \( q \in F[x] \). Then we see \( \deg(x - 1) = \deg(h) + \deg(q) \), so because \( \deg(x - 1) = 1 \) and \( \deg(h) \geq 1 \) we conclude \( \deg(h) = 1 \) and \( \deg(q) = 0 \). But because \( x - 1 \) and \( h \) are both monic we conclude that \( q = 1 \), and thus \( h(x) = x - 1 \), so \( I = \langle x - 1 \rangle \).

4. (Problem 29) Let \( F \) be a field and \( h = pq \) in \( F[x] \), all polynomials monic. If \( p \) and \( q \) are relatively prime in \( F[x] \), show that \( F[x]/\langle h \rangle \cong F[x]/\langle p \rangle \times F[x]/\langle q \rangle \).

**Solution:** Because \( p \) and \( q \) are relatively prime, we have \( \text{lcm}(p, q) = pq = h \) as well as \( \gcd(p, q) = 1 \). Then using Problem 25 we see that

\[
\langle p \rangle \cap \langle q \rangle = \langle \text{lcm}(p, q) \rangle = \langle h \rangle \quad \text{and} \quad \langle p \rangle + \langle q \rangle = \langle \gcd(p, q) \rangle = \langle 1 \rangle = F[x].
\]

The latter equality shows we can invoke the Chinese Remainder Theorem, and doing so we find

\[
F[x]/\langle h \rangle = F[x]/\langle (p) \cap (q) \rangle \cong F[x]/\langle p \rangle \times F[x]/\langle q \rangle.
\]