Part I: from the textbook

Chapter I, Section 12: 1, 3

Part II

In what follows, a ring of rank $n$ is a commutative ring with unit that is also a free $\mathbb{Z}$-module of rank $n$.

1. Prove that a finite integral domain is a field.

2. Prove that (up to isomorphism) the only ring of rank 1 is $\mathbb{Z}$ itself.

3. Prove that an order in a number field of degree $n$ is a ring of rank $n$.

4. Let $S$ be ring of rank 2, also called a quadratic ring.
   (a) Prove that $S$ admits a $\mathbb{Z}$-basis of the form 1, $\tau$.
   (b) For any basis 1, $\tau$ of $S$, specifying the ring structure on $S$ is equivalent to specifying $b, c \in \mathbb{Z}$ such that
       $$\tau^2 = b\tau + c.$$  
       Prove that the quantity $D = b^2 + 4c$ does not depend on the choice of $\tau$. The quantity $D$ is called the discriminant of $S$ and is denoted $\text{disc}(S)$.
   (c) Show that $S$ admits a basis 1, $\tau$ such that
       $$\tau^2 = c \text{ or } \tau^2 = \tau + c$$  
       with $c \in \mathbb{Z}$.
   (d) Conclude that the discriminant of $S$ is congruent to either 0 or 1 modulo 4.

5. Prove that if $K$ is a quadratic number field, the discriminant of $K$ defined in lecture and the discriminant of $\mathcal{O}_K$ as a quadratic ring are the same.
6. Prove that isomorphism classes of quadratic rings $S$ are in canonical bijection with elements of the set

$$\mathcal{D} = \{D \in \mathbb{Z}; D \equiv 0,1 \pmod{4}\}$$

of discriminants. Under this bijection, a quadratic ring $S$ corresponds to $\text{disc}(S) \in \mathcal{D}$, and an element $D \in \mathcal{D}$ corresponds to the quadratic ring

$$S(D) = \mathbb{Z} \left[ \frac{D + \sqrt{D}}{2} \right].$$