MATH 103A – Practice Problems for the Final

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2. Write down 5 different groups of order 24, such that no two of them are isomorphic to each other. Prove that no two are isomorphic.

   Solution. $Z_8 \oplus Z_3, Z_4 \oplus Z_2 \oplus Z_3, Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3, D_{12}, S_4$.

   The first three are abelian, the last two are not. The abelian groups are pairwise not isomorphic because the first one has elements of order 8, the second has no element of order 8, but it does have elements of order 4, while the third does not have elements of either order 8 or 4. For the last two groups, D_{12} has 13 elements of order 2 (12 flips and the rotation by 180°). On the other hand, the elements of order 2 in S_4 are either permutations (6 of them) or product of disjoint permutations (3 elements), for a total of 9 elements. So $D_{12} \not\approx S_4$.

3. Fix some $n \geq 3$, and let $\alpha \in S_n$ be an *odd* permutation. Prove that there must exist an *even* permutation β such that $\alpha = (12)\beta$.

   Solution. Just take $\beta = (12)\alpha$.

4. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 6 & 2 & 4 & 1 & 5 \end{bmatrix}$ be a permutation in S_7.. Find α^{34}, expressing your answer in cycle notation.

   Solution. $\alpha = (136)(2754)$, so $\alpha^{34} = (136)^{34}(2754)^{34}$. The order of (134) is 3, so $(136)^{34} = (134)$. The order of (2754) is 4, so $(2754)^{34} = (2754)^2 = (25)(47)$.

   Hence $\alpha^{34} = (136)(25)(47)$.

5. Suppose that N is a **normal** subgroup of the symmetric group S_5. Show that if $(12) \in N$ then $N = S_5$.

   Solution. If $(12) \in N$, then $\sigma (12) \sigma^{-1} = (\sigma(1), \sigma(2)) \in N$ for all $\sigma \in S_5$. Since $(\sigma(1), \sigma(2))$ may be any two cycle, it follows that $(i, j) \in N$ for all $i \neq j$. As N is a group it must contain all the products of 2- cycles. This implies $\sigma \in N$ for any $\sigma \in S_5$ since every permutation may be written as a product of 2- cycles.

6. Prove that an abelian group with two elements of order 2 must have a subgroup of order 4.

   Solution. Let $a, b \in G$ with $a \neq b$ and $|a| = 2 = |b|$. Then $H := \{e, a, b, ab\}$ is the desired subgroup. Notice that $ab \neq a$ or b by cancellation and $ab \neq e$ since otherwise, $a = b^{-1} = b$. Thus $|H| = 4$.

7. Suppose that H and K are subgroups of a group G. If $|H| = 15$ and $|K| = 28$, find $|H \cap K|$.

   Solution. Since $H \cap K$ is a subgroup of both H and K, it follows by Lagrange's theorem that $|H \cap K|$ must divide both $|H| = 15$ and $|K| = 28$. Since 15 and 28 are relatively prime, we again conclude that $|H \cap K| = 1$.

8. Explain why the function, $\varphi : Z_{12} \to Z_{10}$ defined by $\varphi(x) = x \bmod 10$ is **not** a group homomorphism.

Solution. The point is that this is not a well-defined function. But since the problem asks about a homomorphism, we can make our life easier by using the operation-preserving property. If φ were a homomorphism, then $\varphi(0) = \varphi(12 \cdot 1) = 12 \cdot \varphi(1)$. Yet,

$$12 \cdot \varphi(1) = 12 \bmod 10 = 2 \neq 0 = \varphi(0).$$

9. Describe all of the homomorphisms, φ, from $Z_{12} \to Z_{10}$.

   Solution. To get a homomorphism, we must have $12 \cdot \varphi(1) = 0$ in Z_{10}, that is $10|12 \cdot \varphi(1)$, i.e. $5|\varphi(1)$. Therefore $\varphi(1)$ is either 0 or 5 and hence $\varphi(x) = 0$ in the first case and $\varphi(x) = 5x \bmod 10$ in the second case.

10. Let N be a normal subgroup of a finite group G. Prove that the order of the group element gN in G/N divides the order of g.

    Solution. Indeed, $(gN)^{|g|} = g^{|g|}N = eN$, and thus we know that $|gN|$ divides $|g|$. Alternatively, let $\pi : G \to G/H$ be the homomorphism, $\pi(g) = gH$. Then we know that $|\pi(g)|$ divides $|g|$, which again states $|gN|$ divides $|g|$.

11. How many elements of order 6 are in $S_3 \oplus S_3$? How many cyclic subgroups of order 6 does $S_3 \oplus S_3$ have?

    Solution. S_3 has one element of order 1 (the unit), 3 elements of order 2 (permutations) and 2 elements of order 3 (the two 3-cycles). Hence $|(\sigma, \tau)| = \mathrm{lcm}(|\sigma|, |\tau|) = 6$ only when $|\sigma| = 2, |\tau| = 3$ or viceversa. Each situation gives $3 \cdot 2 = 6$ possibilities. So there are 12 elements of order 6 are in $S_3 \oplus S_3$.

    Each cyclic subgroup of order 6 contains $\phi(6) = 2$ elements of order 6, so there are $12/2 = 6$ cyclic subgroups of order 6 in $S_3 \oplus S_3$.

12. Find 11^{-1} where 11 is thought of as an element of $U(19)$.

    Solution. There are two possibilities.

    First one: use the Euclidean algorithm to get $1 = 7 \cdot 11 - 4 \cdot 19$ and therefore taking this equation mod 19 shows $1 = 7 \cdot 11 \pmod{19}$. Therefore, $7 = 11^{-1}$. As a check we have,

    $$7 \cdot 11 = 77 = 4 \cdot 19 + 1.$$

    Second one: consider $\langle 11 \rangle = \left\{1, 11, 11^2 \pmod{19} = 7, 11^3 = 77 \pmod{19} = 1\right\}$ from which we see that $11^{-1} = 11^2 = 7$.

13. Find all left cosets of $H = \{1, 11\}$ in $G = U(20)$. Find the isomorphism class of G/H.

    Solution.

    $$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$$

    The cosets are given by

    $$H = \{1, 11\}, \ 3H = \{3, 13\}, \ 7H = \{7, 17\}, \text{ and } 9H = \{9, 19\}.$$

    Since these are four distinct cosets and $|U(20) : H| = 8/2 = 4$, we must have all of them. Alternatively observe that there union is all of $U(20)$ and they are distinct (disjoint) cosets.

    This means that the G/H is an abelian group with 4 elements, hence it is isomorphic to either Z_4 or $Z_2 \oplus Z_2$. Now $(3H)(3H) = 9H$, so $3H$ does not have order 2. Thus $G/H \approx Z_4$.

14. Let G be a group of permutations of the set $S = \{1, 2, 3\}$ such that $\mathrm{orb}_G(2) = \{1, 2\}$. Determine G.

Solution. G is a subgroup of S_3 and since $3 \notin \text{orb}_G(2)$ we have that (123) and (23) are not elements of G. Since $(132)^2 = (123)$ this implies $(132) \notin G$. So $G \subset \{\epsilon, (12), (13)\}$. Since $1 \in \text{orb}_G(2)$, it follows that $(12) \in G$.

But if both (12) and (13) were elements of G, then their product would be an element of G. So we would have $(12)(13) = (132) \in G$. And we know that this is not the case. So $(13) \notin G$.

Hence $G = \{\epsilon, (12)\}$.

15. Up to isomorphism, how many abelian groups of order 210 are there? How about order 40?

Solution. If $|G| = 210 = 2 \cdot 3 \cdot 5 \cdot 7$, then

$$G \approx (\text{abelian group of order 2}) \oplus (\text{abelian group of order 3}) \oplus \\ (\text{abelian group of order 5}) \oplus (\text{abelian group of order 7}).$$

But abelian group of order 2 $\approx Z_2$, etc... So $G \approx Z_2 \oplus Z_3 \oplus Z_5 \oplus Z_7 \approx Z_{210}$. There is only one abelian group of order 210 up to isomorphism.

On the other hand, if $|G| = 40 = 2^3 \cdot 5$ then $G \approx$ (abelian group of order 8)$\oplus$ (abelian group of order 3). There is only one possibility for the abelian group of order 3, but there are 3 possibilities for an abelian group of order 8, namely $Z_8, Z_4 \oplus Z_2$ and $Z_2 \oplus Z_2 \oplus Z_2$. So, up to isomorphism, there are $3 \cdot 1 = 3$ abelian groups of order 40.

16. Compute the centralizer of the flip around the horizontal line in D_4.

Solution. See Exam 1, yellow version.

17. What are the possible orders of the elements of $U(72)$?

Solution. $72 = 2^3 \cdot 3^2$, so $U(72) \approx U(8) \oplus U(9) \approx Z_2 \oplus Z_2 \oplus Z_6$. In $Z_2 \oplus Z_2 \oplus Z_6$, the possible orders are

$\|a\|$	$\|b\|$	$\|c\|$	$\|(a,b,c)\|$
1	1	1	1
2	1	1	2
1	2	1	2
2	2	1	2
1	1	2	2
2	1	2	2
1	2	2	2
2	2	2	2
1	1	3	3
2	1	3	6
1	2	3	6
2	2	3	6
1	1	6	6
2	1	6	6
1	2	6	6
2	2	6	6

18. Find the isomorphism class of $U(72)$.

Solution. $72 = 2^3 \cdot 3^2$, so $U(72) \approx U(8) \oplus U(9) \approx Z_2 \oplus Z_2 \oplus Z_6$.