MATH 103A October 20, 2010

Exam 1

Name: SOLUTIONS - blue version.

PID: ______________________

There are 7 pages and 6 questions, for a total of 120 points.

No notes, no calculators, no books.

Please turn off and put away all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question	Points	Score
1	15	
2	20	
3	20	
4	20	
5	35	
6	10	
Total:	120	

1. (15 points) List all the cyclic subgroups of $U(12)$.

$U(12) = \{1, 5, 7, 11\}$

$\langle 1 \rangle = \{1\}$

$\langle 5 \rangle = \{1, 5\}$

$\langle 7 \rangle = \{1, 7\}$

$\langle 11 \rangle = \{1, 11\}$

These are all the cyclic subgroups and no two of them coincide. So the list is complete.

2. (a) (10 points) How many elements of order 7 does $(Z_{21}, + \pmod{21})$ have? Justify your answer.

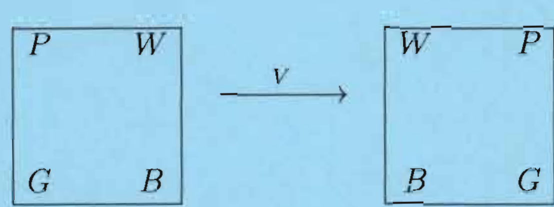
   (b) (10 points) How many elements of order 7 does Z_{2010} have? Justify your answer.

(a) $\left.\begin{array}{l}\mathbb{Z}_{21} \text{ cyclic, order } 21 \\ 7 \mid 21\end{array}\right\} \Rightarrow \mathbb{Z}_{21}$ has $\varphi(7) = 6$ elements of order 7

(b) $\left.\begin{array}{l}\mathbb{Z}_{2010} \text{ cyclic, order } 2010 \\ 7 \nmid 2010\end{array}\right| \Rightarrow$

$\Rightarrow$ there are no elements of order 7 in $\mathbb{Z}_{2010}$.

3. (20 points) Compute the centralizer in D_4 of the flip across the vertical axis.



It might help (**but you don't have to use this if you know a simpler way to solve the problem**) to know that the matrix of a [redacted]clockwise rotation by angle θ is $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, and the matrix of a flip across a line that makes an angle α with the positive x-axis is $S_\alpha = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$.

Similar to yellow version.

Sol II

D_4 consists of matrices

$$R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

If I want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $b = c = 0$

so in general matrices that commute with V are of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Looking at the 8 elements of D_4, these are

$$C(V) = \{ R_0, R_{180}, H, V \}$$

4. (20 points) Suppose that $(G, \cdot)$ is a group and that $x^2 = e$ for all elements $x \in G$. Prove that G is abelian.

See yellow version.

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5. Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix}; a \in \mathbb{R}, a \neq 0 \right\}$.

(a) (10 points) Find an element of G that acts like the identity element in G with respect to matrix multiplication (i.e. find $X \in G$ such that $AX = XA = A$ for all $A \in G$.)

(b) (10 points) Show that G is closed under matrix multiplication (i.e. show that $AB \in G$ for any matrices $A \in G$ and $B \in G$.)

(c) (15 points) Show that (G, matrix multiplication) is a group. You can assume associativity, since we know that matrix multiplication is associative in general.

(a) $\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$

So in order ~~for~~ $\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$

~~for~~ all a, need $2ab = a \Leftrightarrow b = \frac{1}{2}$

Check that $X = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ works on the left as well:

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

So X ~~=~~ $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is the unit in G

(b) $\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$ $\Big| \Rightarrow$

$a \neq 0, b \neq 0 \Rightarrow 2ab \neq 0$

$\Rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} \in G$

(c) All we need are inverses: $\begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$\Rightarrow 2ab = 1/2 \Rightarrow b = 1/4a$ so $\begin{bmatrix} a & a \\ a & a \end{bmatrix}^{-1} = \begin{bmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{bmatrix}$ $\in G$

6. (10 points) Decompose into disjoint cycles the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 1 & 3 & 6 & 2 & 4 \end{bmatrix}.$$

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