

Name: SOLUTIONS (green version)

PID: _____

TA: _____

Section #: _____ Section time: _____

There are 6 pages and 4 questions, for a total of 60 points.

No notes, no calculators, no books.

Please turn off all electronic devices.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

Question:	1	2	3	4	Total
Points:	20	20	10	10	60
Score:					

1. Consider the points in space $P = (2, 3, 1)$, $Q = (2, 1, -1)$, $R = (0, 1, 3)$.

(a) (3 points) Compute the vector \overrightarrow{PQ} .

$$\overrightarrow{PQ} = \langle 0, -2, -2 \rangle$$

(b) (5 points) Find a normal vector to the plane through P, Q and R .

$$\overrightarrow{PR} = \langle -2, -2, 2 \rangle \quad \overrightarrow{QR} = \langle -2, 0, 4 \rangle$$

$$\text{normal vector} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -2 \\ -2 & -2 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -2 \\ -2 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & -2 \\ -2 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} \hat{k}$$

$$= -8\hat{i} + 4\hat{j} - 4\hat{k} = \langle -8, 4, -4 \rangle$$

(c) (6 points) Compute the area of the triangle in space PQR .

$$\text{Area } \Delta = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \sqrt{64+16+16} = \frac{\sqrt{96}}{2} = 2\sqrt{6}$$

(d) (6 points) Find the cosine of the angle at P of the triangle PQR .

$$\cos \theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}| |\overrightarrow{PR}|} = \frac{0+4-4}{\sqrt{8} \cdot \sqrt{12}} = 0$$

2. Consider a moving point whose trajectory is given by $\vec{r}(t) = \left\langle t^2, t^3, 1 - \frac{2}{3}t \right\rangle$.

(a) (3 points) Find the velocity vector $\vec{v}(t)$.

$$\vec{v}(t) = \vec{r}'(t) = \left\langle 2t, 3t^2, -\frac{2}{3} \right\rangle$$

(b) (7 points) Write down a parametric equation for the tangent line $\vec{L}(\theta)$ at the point $P = (1, 1, 1/3)$.

$$1 - \frac{2}{3}t = \frac{1}{3} \quad t = 1$$

$$\vec{v}(1) = \left\langle 2, 3, -\frac{2}{3} \right\rangle$$

$$\begin{aligned} \vec{L}(\theta) &= \left\langle 1, 1, \frac{1}{3} \right\rangle + \theta \left\langle 2, 3, -\frac{2}{3} \right\rangle \\ &= \left\langle 1 + 2\theta, 1 + 3\theta, \frac{1}{3} - \frac{2}{3}\theta \right\rangle \end{aligned}$$

(c) (2 points) What is the speed of the moving point at P ?

$$\begin{aligned} \|\vec{v}(1)\| &= \sqrt{2^2 + 3^2 + \left(\frac{2}{3}\right)^2} = \sqrt{4 + 9 + \frac{4}{9}} \\ &= \sqrt{\frac{36 + 81 + 4}{9}} = \frac{\sqrt{121}}{3} = \frac{11}{3} \end{aligned}$$

(d) (3 points) Find the acceleration $\vec{a}(t)$.

$$\vec{a}(t) = \vec{r}''(t) = \langle 2, 6t, 0 \rangle$$

(e) (5 points) Find the length travelled by the moving point along the trajectory between $t_1 = 0$ and $t_2 = 1$. (Hint: $(a+b)^2 = a^2 + b^2 + 2ab$.)

$$s = \int_0^1 \|\vec{v}(t)\| dt = \int_0^1 \sqrt{4t^2 + 9t^4 + 4/9} dt$$

$$= \int_0^1 \sqrt{(3t^2 + 2/3)^2} dt$$

$$= \int_0^1 (3t^2 + 2/3) dt = t^3 + 2/3 t \Big|_0^1$$

$$= 1 + 2/3 = 5/3$$

$$\begin{aligned} 9t^4 &= b^2 & b &= \sqrt{9t^4} \\ 4/9 &= a^2 & a &= 2/3 \\ 4t^2 &= 2ab \end{aligned}$$

3. Consider the plane $2x + y - z = 0$.

(a) (3 points) Find a normal vector to this plane.

$$n = \langle 2, 1, -1 \rangle$$

(b) (7 points) Find the intersection of this plane with the line parallel to the vector $\langle 1, -1, 1 \rangle$ and passing through the point $P = (1, 1, 1)$.

Eq. of line

$$r(t) = \langle 1, 1, 1 \rangle + t \langle 1, -1, 1 \rangle = \langle 1+t, 1-t, 1+t \rangle$$

$$2(1+t) + (1-t) - (1+t) = 0$$

$$2 = 0 ?$$

which means that plane and line don't intersect.

4. (10 points) Find a parametric equation for the curve $\vec{r}(t)$ given by the intersection of

$$x + 2y^2 = 5 \text{ and } x - 2y^2 + z + 2 = 0.$$

$$\text{Sol I } y = t \Rightarrow x = 5 - 2t^2$$

$$z = -2 - x + 2y^2 = 4t^2 - 7$$

$$\vec{r}(t) = \langle 5 - 2t^2, t, 4t^2 - 7 \rangle$$

$$\text{Sol II } x = t \Rightarrow y^2 = \frac{5-t}{2} \Rightarrow y = \pm \sqrt{\frac{5-t}{2}}$$

$$z = -2 - x + 2y^2 = 3 - 2t$$

$$\vec{r}_1(t) = \langle t, \sqrt{\frac{5-t}{2}}, 3-2t \rangle \text{ AND } \vec{r}_2(t) = \langle t, -\sqrt{\frac{5-t}{2}}, 3-2t \rangle$$

(need 2 functions)

$$\text{Sol III } z = t \Rightarrow x - 2y^2 = -2 - t$$

$$x + 2y^2 = 5$$

$$\hline 2x = 3 - t \Rightarrow x = \frac{3-t}{2}$$

$$y^2 = \frac{1}{2}(5-x) = \frac{t+7}{4} \Rightarrow y = \pm \frac{\sqrt{t+7}}{2}$$

$$\vec{r}_1(t) = \langle \frac{3-t}{2}, \frac{\sqrt{t+7}}{2}, t \rangle \text{ AND } \vec{r}_2(t) = \langle \frac{3-t}{2}, -\frac{\sqrt{t+7}}{2}, t \rangle$$

(need 2 functions)