MATH 20C Lecture 2 - Monday, September 27, 2010

Observations

1. Two vectors pointing in the same direction are scalar multiples of each other.
2. The sum of three head-to-tail vectors in a triangle is 0.

Lines in space

Polled the students about what kind of object is described by the equation $x + 2y = 5$ in the plane. Most got it right (a line). How about the equation $x + 2y + 3z = 7$ in space? It's a plane (about half of the students answered correctly, the other half thought it was a line).

So need some other way to think about lines in space. Think of it as the trajectory of a moving point. Express lines by a parametric equations.

In coordinates. For instance, the line through $Q_0 = (1, 2, 2)$ and $Q_1 = (1, 3, 1)$: moving point $Q(t) = (x, y, z)$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$. Its "velocity" is $\vec{v} = \overrightarrow{Q_0Q_1}$, so $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$. In our example we get $\langle x+1, y-2, z-2\rangle = t\langle 2, 1, -3\rangle$, i.e.

$$\begin{cases} x(t) &= -1 + 2t \\ y(t) &= 2 + t \\ z(t) &= 2 - 3t \end{cases}$$

In vector form. The line through a point P in the direction given by some vector $\vec{v}$ is given by

$$\vec{r}(t) = \overrightarrow{OP} + t\vec{v}$$

where $\vec{r}(t)$ is the position vector of the point $Q(t)$ on the line. Did example with $P = (0, 1, 1)$ and $v = \langle 0, 5, -1\rangle$. (Not sure I remember the exact numbers, I had made it up on the spot in class.)

Dot product

Definition $\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3 + \ldots$ (a scalar, not a vector)

Geometrically $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$, where θ is the angle between the two vectors.

Explained the result as follows. First, $\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}, \vec{B}$ and $\vec{C} = \vec{A} - \vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta$. On the other hand, we get

$$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

Hence the geometric interpretation is a vector formulation of the law of cosines.

MATH 20C Lecture 3 - Wednesday, September 29, 2010

Recall that $\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3 + \ldots = |\vec{A}||\vec{B}| \cos\theta$.

Today: applications of the dot product

1. **Computing lengths and angles (especially angles)**: $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$.

   For instance, in the triangle in space with vertices at $P = (1, 0.0), Q = (0, 1, 0), R = (0, 0, 2)$, the angle θ at P:

   $$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1,1,0\rangle\cdot\langle -1,0,2\rangle}{\sqrt{(-1)^2+1^2+0^2}\ \sqrt{(-1)^2+0^2+2^2}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

2. **Relative direction of two vectors**

   $$\text{sign}(\vec{A}\cdot\vec{B}) = \begin{cases} > 0 & \text{, if } \theta < 90^\circ \text{(acute angle, the two vectors point more or less in the same direction)} \\ = 0 & \text{, if } \theta = 90^\circ \Leftrightarrow \vec{A}\perp\vec{B} \\ < 0 & \text{, if } \theta > 90^\circ \text{(obtuse angle, vectors point away from each other)} \end{cases}$$

3. **Detecting orthogonality:** it's worth emphasizing that $\vec{A}\perp\vec{B} = 0 \Leftrightarrow \vec{A}\cdot\vec{B} = 0$.

4. **Finding the component of a vector in a given direction** If $\vec{u}$ is a unit vector, the component of a vector A in the direction of $\vec{u}$ has length $|\vec{A}|\cos\theta = |\vec{A}||\vec{u}|\cos\theta = \vec{A}\cdot\vec{u}$ ($\theta =$ the angle between the two vectors). The component itself is a vector called the *projection* of $\vec{A}$ in the direction of $\vec{u}$, namely the vector

   $$\vec{A}_{||} = \mathbf{proj}_{\vec{u}}(\vec{A}) = (\vec{A}\cdot\vec{u})\vec{u}.$$

   The component of $\vec{A}$ in the direction perpendicular to $\vec{u}$ is $\vec{A}_\perp = \vec{A} - \vec{A}_{||}$, so

   $$\vec{A} = \vec{A}_{||} + \vec{A}_\perp.$$

   Example: Find the component of $\vec{A} = \langle 1, 2, 3\rangle$ in the direction of the vector $\vec{v} = \langle 1, 1, 0\rangle$.

   Step 1 Find the unit vector in the direction of $\vec{v}$:

   $$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1,1,0\rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

   Step 2 Find the length of the component:

   $$|\vec{A}_{||}| = \vec{A}\cdot\vec{u} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

   Step 3 Multiply the results from step 1 and step 2:

   $$\mathbf{proj}_{\vec{v}}(\vec{A}) = \frac{3}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle = \left\langle \frac{3}{2}, \frac{3}{2}, 0 \right\rangle.$$

   Application: object moving on a frictionless inclined ramp. The physically important quantities are the components of the weight $\vec{F}$ of object (pointing downward) in the direction of the ramp (causes the object to move) and the direction perpendicular to the ramp (creates counter-reaction and keeps the object on the ramp).

5. **Planes**

   The plane $x + 2y + 3z = 0$ consists of the points $P = (x, y, z)$ with the property that $\vec{A} \cdot \overrightarrow{OP} = x + 2y + 3z = 0$, where $\vec{A} = \langle 1, 2, 3 \rangle$. This is the same as saying that $\vec{A} \perp \overrightarrow{OP}$.

Area

We can decompose the area of a polygon in the plane into a sum of areas of triangles. The area of the triangle with sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}$base$\times$height $= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta = (\frac{1}{2}$area of the parallelogram).

So we need to compute $\sin\theta$. We know how to compute $\cos\theta$. Could do $\sin^2\theta + \cos^2\theta = 1$, but get ugly formula. Instead reduce to complementary angle $\theta' = \frac{\pi}{2} - \theta$ by considering $\vec{A}' = \vec{A}$ rotated by $90° = \frac{\pi}{2}$ counterclockwise (drew a picture).

Then, the area of the parallelogram with sides $\vec{A}, \vec{B}$ is $= |\vec{A}||\vec{B}|\sin\theta = |\vec{A}'||\vec{B}|\cos\theta' = \vec{A}' \cdot \vec{B}$

MATH 20C Lecture 4 - Friday, October 1, 2010

Continued from last time: If $\vec{A} = \langle a_1, a_2 \rangle$ and $\vec{A}' = \vec{A}$ rotated by $90° = \frac{\pi}{2}$ counterclockwise, what are the coordinates of $\vec{A}'$? (showed slide, multiple choice).

Answer: $\langle -a_2, a_1 \rangle$ (most students got it right).

So area of the parallelogram with sides $\vec{A}, \vec{B}$ is $= \vec{A}' \cdot \vec{B} = \langle -a_2, a_1 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_2 - a_2 b_1$.

Determinants in the plane

Definition: The determinant of vectors $\vec{A}, \vec{B}$ is $\det\begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$.

Geometrically: $\det\begin{pmatrix} \vec{A} \\ \vec{B} \end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of the parallelogram. (Area is positive, determinant might be negative, so take absolute value.)

Determinants in space

Definition: The determinant of vectors $\vec{A}, \vec{B}, \vec{C}$ is

$$\det\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Geometrically: $\det\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = \pm$ the area of the parellipiped with sides $\vec{A}, \vec{B}, \vec{C}$.

Cross-product

Is defined only for 2 vectors in space. Gives a vector (not a scalar, like dot product).

Definition: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$
(the 3×3 determinant is a *symbolic notation*, the actual formula is the expansion).

Geometrically: $\vec{A}\times\vec{B}$ is a vector with

- length: $|\vec{A}\times\vec{B}|$ = area of the parallelogram with sides $\vec{A}, \vec{B}$;
- direction: perpendicular on the plane containing $\vec{A}, \vec{B}$ and pointing in the direction given by the right hand rule.

Right hand rule:

1. extend right hand in direction of $\vec{A}$
2. curl fingers towards direction of $\vec{B}$
3. thumb points in same direction as $\vec{A}\times\vec{B}$

Question Compute $\hat{\imath}\times\hat{\jmath}$ =? (multiple choice) Answer: $\hat{k}$ (most got it right). Checked both by picture and formula.
Another example: $\vec{A} = \langle 5, 2, -7\rangle$, $\vec{B} = \langle 3, 0, 1\rangle$. Then

$$\vec{A}\times\vec{B} = \begin{vmatrix} 2 & -7 \\ 0 & 1 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 5 & -7 \\ 3 & 1 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 5 & 2 \\ 3 & 0 \end{vmatrix}\hat{k} = 2\hat{\imath} - 26\hat{\jmath} - 6\hat{k} = \langle 2, -26, -6\rangle.$$

(numbers might be slightly different from the ones in class).

Properties of the cross product:

1. $\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$
2. $(2\vec{A})\times(3\vec{B}) = 6(\vec{A}\times\vec{B})$
3. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$
4. $\vec{A}\times\vec{A} = 0$