MATH 20C Lecture 5 - Monday, October 4, 2010

Planes

1. The plane through 3 points, P_1, P_2, P_3.

   A point $P = (x, y, z)$ is in the plane if and only if the volume of the parallelipiped with sides $\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}$ has volume 0 (drew picture). This is the same as saying that

   $$\det \begin{pmatrix} \overrightarrow{P_1P} \\ \overrightarrow{P_1P_2} \\ \overrightarrow{P_1P_3} \end{pmatrix} = 0$$

   Example Take $P_1 = (0, 1, 0), P_2 = (1, 1, 0), P_3 = (1, 0, 0)$. The plane through these 3 points has equation

   $$\det \begin{pmatrix} \langle x, y-1, z\rangle \\ \langle 1, 0, 0\rangle \\ \langle 1, -1, 0\rangle \end{pmatrix} = 0$$

   which is to say $z = 0$. This is the xy-plane.

   Note: In general the equation of a plane in space has the form

   $$\boxed{ax + by + cz = d}$$

2. The plane through the origin perpendicular to $\overrightarrow{N} = \langle 1, 5, 10\rangle$

   Drew a picture. A point $P = (x, y, z)$ is in this plane if and only if $\overrightarrow{OP} \perp \overrightarrow{N}$, which is to say $\overrightarrow{OP} \cdot \overrightarrow{N} = 0$. This means

   $$\langle x, y, z\rangle \cdot \langle 1, 5, 10\rangle = 0,$$

   which gives $x + 5y + 10z = 0$.

3. The plane through $P_0 = (2, 1, -1)$ and perpendicular to $\overrightarrow{N} = \langle 1, 5, 10\rangle$

   Drew a picture. A point $P = (x, y, z)$ is in this plane if and only if $\overrightarrow{P_0P} \perp \overrightarrow{N}$, which is to say $\overrightarrow{P_0P} \cdot \overrightarrow{N} = 0$. This means

   $$\langle x-2, y-1, z+1\rangle \cdot \langle 1, 5, 10\rangle = 0,$$

   which gives $x + 5y + 10z = -3$.

   This plane is parallel to the plane in the previous example. In both cases, the coefficients of x, y, z are the components of the vector $\overrightarrow{N}$.

   In the case of $x + 5y + 10z = -3$ one gets the constant -3 by plugging in the coordinates of the point P_0 in the left hand side.

   Definition A vector perpendicular to a plane $\mathcal{P}$ is called a normal vector to that plane. Note that this is implies that all normal vectors to a given plane are proportional.

   So the coefficients of x, y, z are the components of a normal vector to the plane. Conversely, if the equation of the plane is $ax + by + cz = d$, then $\langle a, b, c, \rangle$ is a normal vector to it.

Relative positions of lines and planes

Two planes:

are either parallel (if their normal vectors are proportional) or they intersect in a line.

Two lines in space:

Here we have 3 possibilities:

1. parallel (so they are in the same plane): same or opposite direction
2. intersect in a point (again they are in the same plane)
3. skew lines (not in the same plane, but no intersection)

A line and a plane:

To figure it out, take the parametric equation of the line and plug into the equation of the plane. Again, 3 possibilities:

1. the line is parallel the plane (the direction of the line and the normal vector to the plane are perpendicular, but the line and the linear system has no solutions)
2. the line is contained in the plane ((the direction of the line and the normal vector to the plane are perpendicular, and the linear system has infinitely many solutions)
3. the line intersects the plane in a point (the linear system has one solution)

Question: The vector $\vec{v} = \langle 1, 2, 1 \rangle$ and the plane $x+y+3z = 5$ are 1) parallel, 2) perpendicular, 3) neither? Answer: parallel (extremly few people got it).

Reason like this. A perpendicular vector would be proportional to the coefficients, namely to the normal vector $\overrightarrow{N} = \langle 1, 1, 3 \rangle$, and $\vec{v}$ clearly is not. On the other hand, $\vec{v}$ being parallel to the plane means that $\vec{v} \perp \langle 1, 1, 3 \rangle$ and, indeed, $\vec{v} \cdot \overrightarrow{N} = 1 + 2 - 3 = 0$.

Example: Line L given by

$$\begin{cases} x = & 1+t \\ y = & 5+2t \\ z = & 7-t \end{cases}$$

has direction $\vec{v}$ and therefore it is either parallel to or contained in the plane $x + y + 3z = 5$. Plugging the expressions for x, y, z as functions of t into the plane equation, we get

$$1 + t + 5 + 2t + 3(7 - t) = 5.$$

This has no solution, so $L \parallel$ plane.

Another example: Line L_2 given by

$$\begin{cases} x = & 1+t \\ y = & 1+2t \\ z = & 1-t \end{cases}$$

has direction $\vec{v}$ and therefore it is either parallel to or contained in the plane $x + y + 3z = 5$. Plugging the expressions for x, y, z as functions of t into the plane equation, we get

$$1 + t + 1 + 2t + 3(1 - t) = 5.$$

This has infinitely many solutions, so L_2 is contained in the plane.

Note If we get just one solution when plugging in the line equations into the plane equation, that means that the line and the plane intersect in one point.

MATH 20C Lecture 6 - Wednesday, October 6, 2010

Showed slides talking about the intersection of the following 3 planes

$$\begin{aligned} x + y + 2z &= 7 \\ 2x + y - z &= 4 \\ x + 2y + 3z &= 3 \end{aligned}$$

They intersect in the point $(15/2, -15/2, 7/2)$.

Note: in order to find a normal vector to a plane, just take the cross product of 2 vectors in the plane.

Parametric Equations

In general, parametric equations are a good way to describe arbitrary motions in plane or space. We have already seen an example of this earlier in the course, namely lines in space. It is convenient to think of trajectories in terms of the position vector $\vec{r}(t)$.

1. $\vec{r}(t) = \langle 1 + t, 2 + t \rangle$ describes a line in the plane in the direction of the vector $\langle 1, 1 \rangle$, through the point $(1, 2)$.
2. $\vec{r}(t) = \langle 1 + t^3, 2 + t^3 \rangle$

   *Question*Does this describe a 1) line? 2) circle? 3) ellipse? (some got it right)

   Answer: line, and in fact the same line as in the previous example.

   To see this, the components are

   $$x = 1 + t^3$$

   $$y = 2 + t^3$$

   Elliminate the parameter t and get $y = x + 1$.

   Beware! The parametric equation is not unique. That is to say, the same curve in plane or space can be described by many different parametric equations.
3. $\vec{r}(t) = \langle 1 + t^2, 2 + t^2 \rangle$ is only a semiline (part of the same line as in the previous two cases, but only points with coordinates at least $(1, 2)$.
4. $\vec{r}(t) = \langle \cos t, \sin t \rangle$ describes a circle in the plane of radius 1, centered at the origin.

Now the other way around, let's find the parametric equation for a given trajectory.

Problem: Find the circle in the plane of radius 5 centered $P = (1, 3)$.

First, the circle of radius 5 centered at the origin has the parametric equation $\langle 5\cos t, 5\sin t\rangle$. In ordered to obtain the desired circle, just translate by $\overrightarrow{OP}$. So we get

$$\vec{r}(t) = \langle 1 + 5\cos t, 3 + 5\sin t\rangle.$$

Student Question: What if we take $\vec{r}(t) = \langle \sin t, \cos t\rangle$? Do we still get the unit circle?

The answer is "yes", but the point moves on the trajectory in the opposite direction (clockwise vs. counterclockwise).

Also, $\vec{r}(t) = \langle \cos(2t), \sin(2t)\rangle$ also describes the same circle, but the point moves on it twice as fast.

Another example: Find a parametric equation for the circle of radius 5 centered at $P = (1, 6, 8)$ lying in a plane parallel to the xz-plane.

We'll follow the same path as before.

Step 1: Write down a parametric equation for the circle centered at the origin, of radius 5, in the xz-plane. Get $\langle 5\cos t, 0, 5\sin t\rangle$.

Step 2 : Translate by the vector $\overrightarrow{OP}$. We get the answer

$$\vec{r}(t) = \langle 1 + 5\cos t, 6, 8 + 5\sin t\rangle.$$

Intersection of surfaces

Another important way to get curves, is by taking the intersection of two surfaces.

We already know an example, namely the intersection of two planes. Going back to the example from the beginning of the lecture, let's find a parametric equation for the intersection of the first 2 planes,

$$\begin{aligned} x + y + 2z &= 7 \\ 2x + y - z &= 4 \end{aligned}$$

We parametrize in terms of $t = x$. That gives

$$\begin{aligned} y + 2z &= 7 - t \\ y - z &= 4 - 2t \end{aligned}$$

Solve and get $y = 5 - \frac{5}{3}t$ and $z = 1 + \frac{1}{3}t$. So our line has direction $\langle 1, -5/3, 1/3\rangle$ and passes through $(0, 5, 1)$.

Another example: Parametrize the intersection of the surfaces

$$\begin{aligned} x^2 - y^2 &= z - 1 \\ x^2 + y^2 &= 4. \end{aligned}$$

We'll do it in two ways.

First way Choose parameter $x = t$. That gives $y^2 = 4 - t^2$ and $z = -3 + 2t^2$. The problem is that when we solve for y we get two different solutions $y = \pm\sqrt{4 - t^2}$. So we need two parametrizations to describe the whole curve:

$$\vec{r}_1(t) = \langle t, \sqrt{4 - t^2}, 2t^2 - 3\rangle$$

and

$$\vec{r}_2(t) = \langle t, -\sqrt{4 - t^2}, 2t^2 - 3\rangle.$$

Second way Parametrize the second curve by $x = 2\cos t, y = 2\sin t$ and plug into the first equation. Get $z = 1 + 4\cos^2 t - 4\sin^2 t = 1 + 4\cos(2t)$, so

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 1 + 4\cos(2t)\rangle.$$

Last thing: demonstrated the cycloid.

MATH 20C Lecture 7 - Friday, October 8, 2010

Last time: position vector $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}[+z(t)\hat{k}]$.

The velocity vector is $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$ in space or $\vec{v}(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle$in the plane.

Note: We take derivatives, limits and integrate vectors **componentwise**.

The cycloid: curve traced by a point on a wheel of radius 1 that is rolling on a flat surface at unit speed. It has equation

$$\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle.$$

The velocity vector is $\vec{v}(t) = \langle 1 - \cos t, \sin t\rangle$. At $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0\rangle$.

The *speed* is defined as the magnitude of the velocity vector. In this case,
$|\vec{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t}$ (smallest at $t = 0, 2\pi, \ldots$, largest at $t = \pi$).

Remark: The speed is $\left|\frac{d\vec{r}}{dt}\right|$ which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Acceleration: $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \vec{r}\,''(t)$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t\rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1\rangle$ is vertical).

Example: The circle $\vec{r}(t) = \langle \cos t, \sin t\rangle$.

The velocity vector is $\vec{v}(t) = \langle -\sin t, \cos t\rangle$, and the speed is $|\vec{v}| = 1$ for any t. The acceleration vector is $\vec{a} = \langle -\cos t, -\sin t\rangle$.

The tangent line at time t_0 to the trajectory $\vec{r}(t)$ is the line through the point $P = (x(t_0), y(t_0), z(t_0))$ in the direction given by the velocity vector at that point, $\vec{v}(t_0)$. It has parametric equation

$$\vec{L}(s) = \vec{r}(t_0) + s\vec{v}(t_0).$$

(I used parameter s here in order to distinguish it from the parameter t of the trajectory).

Rules

$$\begin{aligned}
\frac{d(\vec{r}_1 \cdot \vec{r}_2)}{dt} &= \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 + r_1 \cdot \frac{d\vec{r}_2}{dt} \\
\frac{d(\vec{r}_1 \times \vec{r}_2)}{dt} &= \frac{d\vec{r}_1}{dt} \times \vec{r}_2 + r_1 \times \frac{d\vec{r}_2}{dt} \text{ (make sure you keep the vectors in the given order!)} \\
\frac{d}{dt}\left[\vec{r}(f(t))\right] &= \left[\vec{r}\,'(f(t))\right] f'(t) \qquad \text{(chain rule)}
\end{aligned}$$

Did exercises 23 and 24 from Section 13.2 (Rogawski's book) as examples for the first two product rules.

Example $\vec{r}(t) = \langle t, e^{2t}, \cos t\rangle$, $f(t) = 5t$

Then $\vec{r}(f(t)) = \langle 5t, e^{10t}, \cos(5t)\rangle$ and $\vec{r}\,'(t) = \langle 1, 2e^{2t}, -\sin t\rangle$. So

$$\vec{r}\,'(f(t)) = \langle 1, 2e^{10t}, -\sin(5t)\rangle,$$

and the chain rule says that

$$\frac{d\langle 5t, e^{10t}, \cos(5t)\rangle}{dt} = f'(t)\langle 1, 2e^{10t}, -\sin(5t)\rangle = 5\langle 1, 2e^{10t}, -\sin(5t)\rangle.$$

It checks out!

Integration

$$\int \vec{r}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle + \vec{c}, \qquad \vec{c} = \langle c_1, c_2, c_3\rangle$$

Going back to $\vec{r}(t) = \langle t, e^{2t}, \cos t\rangle$, we get

$$\int \vec{r}(t)dt = \left\langle \frac{t^2}{2}, \frac{e^{2t}}{2}, \sin t \right\rangle + \langle c_1, c_2, c_3\rangle.$$

Also,

$$\int_0^\pi \vec{r}(t)dt = \left\langle \left.\frac{t^2}{2}\right|_{t=0}^{t=\pi}, \left.\frac{e^{2t}}{2}\right|_{t=0}^{t=\pi}, \sin t\big|_{t=0}^{t=\pi} \right\rangle = \left\langle \frac{\pi^2}{2}, \frac{e^{2\pi}-1}{2}, 0 \right\rangle.$$