MATH 20C Lecture 8 - Monday, October 11, 2010

A bit more about tangent lines. The parameter for the tangent line is different from the parameter of the curve itself.

Take for instance the spiral $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$. The velocity vector is $\vec{v}(t) = \langle -3\sin t, 3\cos t, 4\rangle$. At time π we have $\vec{v}(\pi) = \langle 0, -3, 4\rangle$ and $\vec{r}(\pi) = \langle -3, 0, 4\pi\rangle$. Therefore the tangent line at $t_0 = \pi$ has equation

$$\vec{L}(\theta) = \langle -3, 0, 4\pi\rangle + \theta\langle 0, -3, 4\rangle = \langle -3, -3\theta, 4\pi + 4\theta\rangle.$$

Note that at $\theta = 0$ the line touches the curve, and then it goes away from it.

Arc length

s = distance travelled along trajectory. Since the rate of change of the distance is the speed, we have $\frac{ds}{dt}$ = speed = $|\vec{v}(t)|$. Can recover length of trajectory by integrating ds/dt. So the length of the curve starting at time t_1 until time t_2 is

$$s = \int_{t_1}^{t_2} |\vec{v}(t)| dt.$$

The distance traveled by a moving point along a curve starting at time t_1 until the current time is

$$s(t) = \int_{t_1} |\vec{v}(u)| du.$$

Computing such an integral is not always easy... Sometimes it can be done, though. Here are a few examples.

1. $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t\rangle$

   Then the velocity is $\vec{v} = \langle -4\sin t, 4\cos t, 3\rangle$ and the arc length starting at $t_1 = 0$ until the current time is

   $$s(t) = \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u + 9} du = \int_0^t 5 du = 5u|_0^t = 5t$$

2. $\vec{r}(t) = 2t\hat{\imath} + (\ln t)\hat{\jmath} + t^2\hat{k}$

   Then $\vec{v}(t) = 2\hat{\imath} + \frac{1}{t}\hat{\jmath} + 2t\hat{k}$, and the arc length starting at $t_1 = 1$ is

   $$s(t) = \int_1^t \sqrt{4 + \frac{1}{u^2} + 4u^2} du = \int_1^t \sqrt{(2u + \frac{1}{u})^2} du = \int_1^t (2u + \frac{1}{u}) du = t^2 - 1 + \ln t.$$

   The tangent line at $t_1 = 1$ passes through the point $(2, 0, 1)$ and has direction given by $\vec{v}(1) = \langle 2, 1, 0\rangle$. Therefore it has parametric equation

   $$\vec{L}(\theta) = \langle 2, 0, 1\rangle + \theta\langle 2, 1, 0\rangle = \langle 2 + 2\theta, \theta, 1\rangle.$$

MATH 20C Lecture 9 - Wednesday, October 13, 2010

Example 1 Set up an integral, but do not evaluate, for the arc length starting at time 5 of the curve $\vec{r}(t) = \langle \frac{2}{3}t^3, 4t, -\sqrt{2}t^2 \rangle$.

We have $\vec{v}(t) = \langle 2t^2, 4, -2\sqrt{2}t \rangle$, and

$$s(t) = \int_5^t \sqrt{4t^4 + 16 + 8t^2}.$$

Example 2 $\vec{a}(t) = \hat{k}, \vec{v}(0) = \hat{\imath}, \vec{r}(0) = \hat{\jmath}$. Find $\vec{r}(t)$.

Since $\vec{a}(t) = \frac{d\vec{v}}{dt}$, that means that $\vec{v}(t) = t\hat{k} + \vec{c}$. And now $\vec{v}(0) = \hat{\imath}$, tells us $\vec{c} = \hat{\imath}$. Therefore

$$\vec{v}(t) = t\hat{k} + \hat{\imath}.$$

Repeating the procedure for $\vec{v}(t) = \frac{d\vec{r}}{dt}$, we get

$$\vec{r}(t) = \frac{t^2}{2}\hat{k} + t\hat{\imath} + \hat{\jmath} = \langle t, 1, t^2/2 \rangle.$$

Example 3 $\vec{a}(t) = 2\hat{\imath} + 2t\hat{\jmath}, \vec{v}(0) = 7\hat{\imath}, \vec{r}(0) = 2\hat{\imath} + 9\hat{k}$. Find $\vec{r}(t)$.

Again, $\vec{v}(t) = \int \vec{a}(t)dt = 2t\hat{\imath} + t^2\hat{\jmath} + \vec{c}$. Using the initial condition $\vec{v}(0) = 7\hat{\imath}$ gives

$$\vec{v}(t) = (2t + 7)\hat{\imath} + t^2\hat{\jmath}.$$

Integrate and obtain $\vec{r}(t) = \int \vec{v}(t)dt = (t^2 + 7t)\hat{\imath} + \frac{t^3}{3}\hat{\jmath} + \vec{d}$. The initial condition implies $\vec{d} = \vec{r}(0) = 2\hat{\imath} + 9\hat{k}$, so

$$\vec{r}(t) = (t^2 + 7t + 2)\hat{\imath} + \frac{t^3}{3}\hat{\jmath} + 9\hat{k}.$$

Example 4 A planet is circling a star counterclockwise describing a circle of radius 5. It makes 2 complete revolutions around the star each year. Parametrize the trajectory of the planet as function of t = time in years.

Since the planet rotates twice around the star in a year, that means that in a year it covers twice the circumference of the circle 4π times the radius. So

$$\vec{r}(t) = \langle 5\cos(4\pi t), 5\sin(4\pi t) \rangle.$$

Example 4' How about rotating clockwise and it takes 3 years for a complete revolution?

$$\vec{r}(t) = \langle 5\sin(\frac{2\pi}{3}t), \cos(\frac{2\pi}{3}t) \rangle.$$

Example 5 Parametrize the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Take $\frac{x}{a} = \cos t$ and $\frac{y}{b} = \sin t$. We get

$$\vec{r}(t) = \langle a\cos t, b\sin t \rangle.$$

We ended with Example 4 on page 773 of the textbook.

MATH 20C Lecture 10 - Friday, October 15, 2010

Review

We went through the extra problems in the study guide. And we computed the volume of the parallelipiped with sides $\overrightarrow{P_0P_1}, \overrightarrow{P_0P_2}$ and $\overrightarrow{P_0R}$ where P_0, P_1, P_2 are the points in problem 2 from the study guide and $R = (0, 0, 0)$.

There was also a question about Example 4, page 773 in the book. Unfortunately, due to time constraints, we didn't get to it. Please read that in the book.