MATH 20C Lecture 16 - Monday, November 1, 2010

Implicit differentiation

Example: $x^2 + yz + z^3 = 8$. Viewing $z = z(x, y)$, compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Take $\frac{\partial}{\partial x}$ of both sides of $x^2 + yz + z^3 = 8$. Get $2x + y\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0$, hence $\frac{\partial z}{\partial x} = -\frac{2x}{y+3z^2} = -\frac{2}{3}$.

In general, consider a surface $F(x, y, z) = c$. The we can view $z = z(x, y)$ as a function of two independent variables x, y and compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. To do so, we take the partial derivative with respect to x of both sides of the equation $F(x, y, z) = c$ and get (by the chain rule)

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

But $\partial x/\partial x = 1$ and, since x and y are independent, $\partial y/\partial x = 0$ (changing x does not affect y). Hence the equation above really says that $F_x + F_z\frac{\partial z}{\partial x} = 0$ which implies

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.}$$

Similarly,

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.}$$

Changing gears, let's see how we can recover f from its gradient. Say $\nabla f = \langle 3x^2y, x^3 + 2z, 2y + \cos z\rangle$. We proceed by successive integration. We are given that $f_x = 3x^2y$. Integrating with respect to x (view y, z as constants), we see that $f = x^3y + g(y, z)$. Therefore

$$f_y = x^3 + \frac{\partial g}{\partial y}.$$

But we know from the gradient that $f_y = x^3 + 2z$, hence $g_y = 2z$. Integrate with respect to y and get $g = 2yz + h(z)$, hence $f = x^3y + 2yz + h(z)$. Since $f_z = 2y + \cos z$ we get that $\frac{dh}{dz} = \cos z$, so $h(z) = \sin z + C$. Substituting in the expression of f gives $f = x^3y + 2yz + \sin z + C$.

Min/max in several variables

At a local max or min, $f_x = 0$ and $f_y = 0$ (since (x_0, y_0) is a local max or min of the slice). Because 2 lines determine tangent plane, this is enough to ensure that the tangent plane is horizontal.
Definition A critical point of f is a point (x_0, y_0) where $f_x = 0$ and $f_y = 0$.

A critical point may be a local min, local max, or saddle. Or degenerate.
Example: $f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y$. Critical point: $f_x = 2x - 2y + 2 = 0$, $f_y = 2x + 6y - 2 = 0$, gives $(x_0, y_0) = (-1, 0)$ (only one critical point).

A critical point may be a local min, local max, or saddle. Or degenerate. Pictures shown of each type. To decide, apply **second derivative test**.

Definition The hessian matrix of f is

$$H(x,y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

Second derivative test

Let (x_0, y_0) be a critical point of f.

Case 1 $\det H > 0, f_{xx} > 0$: (x_0, y_0) is a local minimum

Case 2 $\det H > 0, f_{xx} < 0$: (x_0, y_0) is a local maximum

Case 3 $\det H < 0$: (x_0, y_0) is a saddle point

Case 4 $\det H = 0$: cannot tell (need higher order derivatives)

MATH 20C Lecture 17 - Wednesday, November 3, 2010

Example 1 Find the local min/max of $f(x,y) = x + y + \frac{1}{xy}$.

Step 1 *Find critical points by solving the* 2×2 *system of equations*

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

In this case, the system is

$$\begin{cases} \frac{1}{x^2 y} = 1 \\ \frac{1}{xy^2} = 1. \end{cases}$$

Divide the first equation by the second and get $x = y$, plug back into the first equation and get $x^3 = 1$. So the only critical point is $(1,1)$.

Showed slide asking students if this point is a local max/min or saddle. Most got it right (local min). Now let's do it rigorously.

Step 2 *Compute the Hessian matrix*

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

Recall that $f_{xy} = f_{yx}$.

In our case, get $H(x,y) = \begin{bmatrix} \frac{2}{x^3 y} & \frac{1}{x^2 y^2} \\ \frac{1}{x^2 y^2} & \frac{2}{xy^3} \end{bmatrix}$.

Step 2 *Compute the Hessian matrix at each of the critical points.*

$$H(1,1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Step 4 *Apply the second derivative test for each critical point.*

det $H(1,1) = 4 - 1 = 3 > 0$ and $f_{xx} = 2 > 0$, so $(1,1)$ is a local minimum.

Attention! We can also infer the nature of a critical point from the contour plot. Showed picture and discussed possibilities. Most students got the right answer.

Example 2 $f(x,y) = (x^2 + y^2)e^{-x}$

Step 1 *Find critical points by solving the* 2×2 *system of equations*

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

In this case, the system is

$$\begin{cases} (2x - x^2 - y^2)e^{-x} = 0 \\ 2ye^{-x} = 0. \end{cases}$$

The second equation tells us that $y = 0$. Plug back into the first equation and get $x^2 - 2x = 0$. So critical points are $(0,0)$ and $(2,0)$.

Step 2 *Compute the Hessian matrix*

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

In our case, get $H(x,y) = \begin{bmatrix} (2 - 4x + x^2 + y^2)e^{-x} & -2ye^{-x} \\ -2ye^{-x} & 2e^{-x} \end{bmatrix}$.

Step 2 *Compute the Hessian matrix at each of the critical points.*

$$H(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$H(2,0) = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{bmatrix}.$$

Step 4 *Apply the second derivative test for each critical point.*

- For $(0,0)$: det $H(0,0) = 4 > 0$ and $f_{xx} = 2 > 0$, so $(0,0)$ is a local minimum.
- For $(2,0)$: det $H(2,0) = -4e^{-4} < 0$, so $(2,0)$ is a saddle point.

NOTE: the global min/max of a function is not necessarily at a critical point! Need to check boundary / infinity.

In Example 2 above, to find the global min/max of f in the region $0 \leq x, y \leq 1$, w need to check what happens on the boundary. Namely we have to look at $f(0,y), f(1,y), f(x,0)$ and $f(x,1)$. We have to compute the min/max for these 4 functions and compare to the value at critical points inside the square (if any).

MATH 20C Lecture 18 - Friday, November 5, 2010

Least squares method

Set up problem: given experimental data (x_i, y_i) $(i = 1, \dots, n)$, want to find a best-fit line $y = ax+b$ (the unknowns here are a, b, not x, y!) Deviations: $y_i - (ax_i + b)$; want to minimize the total square deviation $D(a,b) = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$.

$\frac{\partial D}{\partial a} = 0$ and $\frac{\partial D}{\partial b} = 0$ leads to a 2×2 linear system for a and b

$$\begin{aligned} \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b &= \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) a + nb &= \sum_{i=1}^{n} y_i \end{aligned}$$

The least-squares setup also works in other cases: e.g. exponential laws $y = ce^{ax}$ (taking logarithms: $\ln y = ax + \ln c$, so setting $b = \ln c$ we reduce to linear case); or quadratic laws $y = ax^2 + bx + c$ (minimizing total square deviation leads to a 3×3 linear system for a, b, c). Example: Moores Law (number of transistors on a computer chip increases exponentially with time): showed picture.

Lagrange multipliers

Problem: min/max of a function $f(x,y,z)$ when variables are constrained by an equation $g(x,y,z) = c$.

Example: find point of hyperbola $xy = 5$ closest to origin. I.e. minimize $\sqrt{x^2 + y^2}$, or better $f(x,y) = x^2 + y^2$, subject to $g(x,y) = xy = 5$. Drawn picture.

Observe on picture: at the minimum, the level curves are tangent to each other, so the normal vectors ∇f and ∇g are parallel.

So: there exists λ ("multiplier") such that $\nabla f = \lambda \nabla g$.

We replace the constrained min/max problem in 2 variables with 3 equations involving 3 variables x, y, λ :

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = c \end{cases} \qquad \text{i.e. in our case} \qquad \begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 5. \end{cases}$$

Substituting the second equation in the first we get $x(\lambda^2/4 - 1) = 0$, so either $x = 0$ or $\lambda = \pm 2$. But if $x = 0$, then $y = 0$ and the constraint $xy = 5$ is not satisfied. Hence we are forced to have $\lambda = \pm 2$. No solutions for $\lambda = -2$, but $\lambda = 2$ gives $(\sqrt{5}, \sqrt{5})$ and $(-\sqrt{5}, -\sqrt{5})$.

Warning: method doesn't say whether we have a min or a max, and second derivative test DOES NOT apply with constrained variables. Need to answer using geometric argument or by comparing values of f.