MATH 20C Lecture 19 - Monday, November 8, 2010

Lagrange multipliers continued

Example: Find the min/max of $f(x, y, z) = 3x + y + 4z$ on the surface $x^2 + 3y^2 + 6z^2 = 1$.

Step 1 *Compute the two gradients ∇f and ∇g.*

$\nabla f = \langle 3, 1, 4\rangle \qquad \nabla g = \langle 2x, 6y, 12z\rangle$

Step 2 *Write down the Lagrange multiplier equations $\nabla f = \lambda \nabla g$ and the constraint $g = c$.*

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad \begin{cases} 3 &= 2\lambda x \\ 1 &= 6\lambda y \\ 4 &= 12\lambda z \end{cases}$$

$$g = c \quad \Longrightarrow \quad x^2 + 3y^2 + 6z^2 = 1$$

Step 3 *Solve the system, i.e. find points (x, y, z) that satisfy the equations from Step 2.*

WARNING! There is no general method to solve these equations. In each case, you have to think about them and come up with a method. Sometimes it will be impossible to solve without using a computer. (Not on the exam though!)

In our example, note that λ cannot be 0. From the first three equations get $x = \frac{3}{2\lambda}, y = \frac{1}{6\lambda}, z = \frac{1}{3\lambda}$. Substitute these values in the constraint equation and get $\lambda^2 = 3$, so $\lambda = \pm\sqrt{3}$. Therefore there are two points $\left(\frac{\sqrt{3}}{2}, \frac{1}{6\sqrt{3}}, \frac{1}{3\sqrt{3}}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{6\sqrt{3}}, -\frac{1}{3\sqrt{3}}\right)$ on the given surface at which the gradients of f and g are parallel.

Step 4 *Plug the points you found into f and compare values.*

WARNING! There is no general method to tell you if the points you got are local min/max/saddle. The only way is to plug in points *on the surface/curve* $g = c$ and compare the values of f.

However, sometimes we can find some geometric reason for which the function f would be forced to have a min or a max on the surface/curve $g = c$. For instance, if $g = c$ is closed, then f will have both min and max along $g = c$.

This happens to be the case for us, since $x^2 + 3y^2 + 6z^2 = 1$ describes the shell of an ovoid in 3-space.

$$f\left(\frac{\sqrt{3}}{2}, \frac{1}{6\sqrt{3}}, \frac{1}{3\sqrt{3}}\right) = 3\frac{\sqrt{3}}{2} + \frac{1}{6\sqrt{3}} + 4\frac{1}{3\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

$$f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{6\sqrt{3}}, -\frac{1}{3\sqrt{3}}\right) = -3\frac{\sqrt{3}}{2} - \frac{1}{6\sqrt{3}} - 4\frac{1}{3\sqrt{3}} = -\frac{2}{\sqrt{3}}.$$

The first point is a max, the second is a min.

Partial differential equations

These are equations involving partial derivatives of an unknown function. They are very important in physics. E.g., heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

describes the evolution of temperature over time.

Review topics

- Functions of several variables, contour plots.
- Partial derivatives, gradient; approximation formulas, tangent planes, directional derivatives.
- higher order partial derivatives
- chain rule, change of variables, implicit differentiation
- to be continued

MATH 20C Lecture 20 - Wednesday, November 10, 2010

Review topics continued: optimization

- Min/max problems: critical points, second derivative test, checking boundary. (least squares won't be on the exam)
- Min/max for non-independent variables: Lagrange multipliers

Went through problems 7–9 and 1 from the suggested practice problems in the study guide. Solutions to all the practice problems are posted online.

MATH 20C Lecture 21 - Friday, November 12, 2010: second midterm