MATH 20E – Final Exam Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the online notes, the relevant sections of the book and the homework problems. If you still have trouble with some of the topics encountered so far, take the book (or another calculus book) and solve more problems related to that topic until you *really* understand how and why things work. The final exam covers everything we discussed throughout the term.

Topics from MATH 20C - should be reviewed as well

vectors; operation with vectors (addition, subtraction, multiplication by scalars, dot product, cross product); determinants, area, volume
calculus with vectors: limits, differentiation (product rules, chain rule), integration
functions of several variables: graphs, level curves and surfaces, contour maps; limits, partial derivatives, gradient, chain rule
double integrals (rectangular and polar coordinates); triple integrals (rectangular and cylindrical coordinates); pay special attention to how you setup the integration

Topics from MATH 20E

functions of several variables, partial derivatives, chain rule, approximation formula, tangent planes; derivative matrix;
integration in several variables, change of variables
surface area, tangent planes to surfaces
vector fields, flow lines, line integrals, work, flux (both 2D and 3D)
Green's theorem
gradient and conservative vector fields (both 2D and 3D)
curl and div, Green's theorem (scalar form, vector form, divergence theorem in the plane)
vector curl, Stokes' theorem, divergence theorem

Concept review

1. Make sure you know how to draw the level curves and the contour map of a function.
2. What are the first order partial derivatives of a function of 2 variables? What about a function of 3 variables? What do they represent geometrically? How do you compute them?
3. Chain rule (all its forms).
4. What are the higher order partial derivatives of a function?
5. Compute $\frac{\partial^3 f}{\partial x \partial z^2}$ and $\frac{\partial^3 f}{\partial x \partial y \partial z}$ of $f(x, y, z) = xe^{-yz}$.
6. What is the gradient of a function?

7. Compute the gradient of $f(x, y, z) = xe^{-y^2 z}$.
8. Write down the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0)$.
9. What does a double/triple integral represent geometrically?
10. More double/triple integrals: make sure you can draw the picture of the region, take slices to set up the iterated integral. And make sure you can interchange the order of integration.
11. Make sure you can set up double integrals in polar coordinates; recall that $dA = rdrd\theta$.
12. Make sure you can set up triple integrals in cylindrical coordinates; recall that $dV = rdzdrd\theta$.
13. Make sure you can set up triple integrals in spherical coordinates; recall that $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.
14. Make sure you can compute the Jacobian of a general change of variables.
15. Make sure you understand how to set up an integral after a change of variables (bounds and integrand).
16. Chain rule (all its forms).
17. What is the gradient of a function?
18. Compute the gradient of $f(x, y, z) = xe^{-y^2 z}$.
19. Write down the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0)$.
20. Make sure you can set up double integrals in polar coordinates; recall that $dA = rdrd\theta$.
21. Make sure you can set up triple integrals in cylindrical coordinates; recall that $dV = rdzdrd\theta$.
22. Make sure you can set up triple integrals in spherical coordinates; recall that $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.
23. Make sure you can compute the Jacobian of a general change of variables.
24. Make sure you understand how to set up an integral after a change of variables (bounds and integrand).
25. What is a vector field in $\mathbb{R}^n$?
26. Make sure you can give a couple of examples from the real world.
27. Make sure you can sketch a vector field in the plane.
28. What is a flow line?
29. Is $\vec{c}(t) = (t, \sin t)$ a flow line for $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$?
30. What is a gradient field? What is its potential?
31. When is a vector field conservative?
32. How do you test to see if a vector field is a gradient field?
33. What is a simply connected region in 2D?
34. What is a simply connected region in space?
35. How do you find the potential of a gradient field? You should remember both methods (antiderivatives and integral along a simple path).
36. Find the potential of the vector field $\vec{F} = (2x + y)\hat{\mathbf{i}} + (x + 2y)\hat{\mathbf{j}}$.

37. What is the work done by a vector field along a trajectory?
38. Make sure you can parametrize curves (both in the plane and the space) and surfaces.
39. Make sure you can set up line and surface integrals, both for scalar functions and vector fields.
40. Make sure you can compute the tangent plane to a surface in the case of graphs, parametric surfaces, implicit surfaces.
41. Make sure you can find a normal vector to a surface in the case of graphs, parametric surfaces, implicit surfaces.
42. Make sure you can compute the surface area for graphs, parametric surfaces, implicit surfaces and surfaces of revolution.
43. Green's theorem (all 3 forms) and its application to area.
44. Stokes' theorem: make sure you understand the orientation.
45. Divergence theorem: the normal vector should always point outwards.

Practice problems

This is a collection of extra problems to help you prepare for the final. Once again, the list is not complete and these problems do **not** imply anything about the content of the exam.

1. Go over HW.
2. Let $f(x, y) = xy - x^4$.
   (a) Find the gradient of f at $(1, 1)$.
   (b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.
3. Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.
4. Let $f(x, y) = x^2y^2 - x$.
   (a) Find ∇f at $(2, 1)$.
   (b) Write the equation for the tangent plane to the graph of f at $(2, 1, 2)$.
   (c) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.
5. Let $u = y/x$, $v = x^2 + y^2$, $w = w(u, v)$.
   (a) Express the partial derivatives w_x and w_y in terms of x, y, w_u and w_v.
   (b) Express $xw_x + yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.
   (c) Find $xw_x + yw_y$ in case $w = v^5$.
6. Set up an iterated integral giving the volume of the solid in space in the shape of the region bounded by the elliptical cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the sphere of radius 4 centered at $(0, 0, 2)$ in the first octant $x, y, z \leq 0$. Give the integrand and bounds, but DO NOT EVALUATE.
7. Find the volume of the region defined by $4 - x^2 - y^2 \leq z \leq 10 - 4x^2 - 4y^2$.

8. Sketch the region of integration and evaluate by changing to polar coordinates

$$\int_0^2 \int_x^{x\sqrt{3}} x\,dy\,dx.$$

9. (a) Draw a picture of the region of integration of $\displaystyle\int_0^1 \int_x^{2x} dydx$.

   (b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$. Warning: your answer will have two pieces.

10. Set up a triple integral to compute the volume of the ball of radius 5. Now evaluate the integral using spherical coordinates.

11. Evaluate

$$\int_0^{1/2} \int_{\sqrt{2y-y^2}}^{\sqrt{1-y^2}} xe^y\,dxdy$$

by changing variables to $u = x^2 + y^2$ and $v = x^2 + y^2 - 2y$. (First you should draw a picture of the region of integration.)

12. Consider the region R in the first quadrant bounded by the curves $y = x^2, y = x^2/5, xy = 2$, and $xy = 4$.

   (a) Compute $dxdy$ in terms of $dudv$ if $u = x^2/y$ and $v = xy$.

   (b) Find a double integral in uv coordinates for the area of R and evaluate it.

13. Sketch the following vector fields:

   (a) $\vec{F}(x, y) = (x, -y)$

   (b) $\vec{F}(x, y) = (y, y^2)$

14. In which of the following cases is $\vec{c}(t)$ a flow line for the vector field $\vec{F}$?

   (a) $\vec{F}(x_1, x_2, x_3, x_4) = \left(3x_3^2, \dfrac{1}{2x_2}, \dfrac{x_1}{x_3^3}, \dfrac{1}{x_3}\right)$ $\quad\vec{c}(t) = (t^3, \sqrt{t}, -t, -\log t)$ for $t > 0$.

   (b) $\vec{F}(x, y) = (-y, x)$ $\quad\vec{c}(t) = (\cos t, \sin t)$.

   (c) $\vec{F}(x, y) = (-y, x)$ $\quad\vec{c}(t) = (\sin t, \cos t)$.

   (d) $\vec{F}(x, y) = (-y, x)$ $\quad\vec{c}(t) = (t, \sin t)$.

15. Determine a, b such that $\vec{c}(t) = (te^{at}, t, e^{2t})$ is a flow line for the vector field $\vec{F} = (z + 2yz)\hat{\mathbf{i}} + b\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$.

16. Find the potential for the following vector fields or show that the potential does not exist.

   (a) $\vec{F}(x, y) = (x, -y)$

   (b) $\vec{F}(x, y) = (y, y^2)$

   (c) $\vec{F}(x, y) = (2xy, x^2 + y^2)$

17. Let $\vec{F} = (ax^2y + y^3 + 1)\hat{\mathbf{i}} + (2x^3 + bxy^2 + 2)\hat{\mathbf{j}}$ be a vector field, where a and b are constants.

   (a) Find the values of a and b for which $\vec{F}$ is a gradient field.

   (b) For these values of a and b, use a integration on a curve to find a potential for $\vec{F}$.

(c) Still using the values of a and b from part (a), compute $\int_C \vec{F} \cdot d\vec{r}$ along the curve C such that $x = e^t \cos t, y = e^t \sin t, 0 \le t \le \pi$.

18. (a) Show that $\vec{F} = (3x^2 - 6y^2)\hat{\mathbf{\imath}} + (-12xy + 4y)\hat{\mathbf{\jmath}}$ is conservative.
    (b) Find a potential function for $\vec{F}$.
    (c) Let C be the curve $x = 1 + y^3(1-y)^3, 0 \le y \le 1$. Calculate $\int_C \vec{F} \cdot d\vec{r}$.

19. For $\vec{F} = yx^3\hat{\mathbf{\imath}} + y^2\hat{\mathbf{\jmath}}$ find the work done by the $\vec{F}$ on the portion of the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

20. (a) Express the work done by the force field $\vec{F} = (5x + 3y)\hat{\mathbf{\imath}} + (1 + \cos y)\hat{\mathbf{\jmath}}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\displaystyle\int_a^b f(t)dt$.(Do not evaluate the integral; don't even simplify $f(t)$.)
    (b) Evaluate the line integral using Green's theorem.

21. Find the surface area of
    (a) the surface obtained by rotating the curve $y = x^{1/2}, 0 \le x \le 1$ about the y-axis.
    (b) the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$.
    (c) S is the surface described by $x = u - v, y = u + v, z = uv$ as u, v are in the disk of radius 2 centered at the origin in the uv-plane.

22. Exercise 9, page 472.

23. Find the tangent plane to the surface S at the point P in the following cases.
    (a) $S : x^2 + y^5 \cos(xy) - z^6 = -648, P = (9,0,3)$.
    (b) S is the graph of $f(x,y) = e^{2x^2 y} + \sin(x+y), P = (-1,1,f(-1,1))$.
    (c) S is the cylinder of radius 5 centered around the y-axis and $P = (3,19,4)$.

24. Find the area of the following plane regions.
    (a) $x^{2/5} + y^{2/5} < 4$;
    (b) one loop of the four-leafed rose $r = 3\sin 2\theta$.

25. Consider the rectangle R with vertices $(0,0), (1,0), (1,4)$ and $(0,4)$. The boundary of R is the curve C, consisting of C_1, the segment from $(0,0)$ to $(1,0)$; C_2, the segment from $(1,0)$ to $(1,4)$; C_3 the segment from $(1,4)$ to $(0,4)$; and C_4 the segment from $(0,4)$ to $(0,0)$. Consider the vector field $\vec{F} = (xy + \sin x \cos y)\hat{\mathbf{\imath}} - (\cos x \sin y)\hat{\mathbf{\jmath}}$.
    (a) For each of C_i find a normal unit $\hat{\mathbf{n}}_i$ vector pointing out of the rectangle.
    (b) Find the flux of $\displaystyle\sum_{i=1}^{4} \int_{C_i} \vec{F} \cdot \hat{\mathbf{n}}_i ds$ out of R through C. Show your reasoning.
    (c) Is the total flux $\displaystyle\sum_{i=1}^{3} \int_{C_i} \vec{F} \cdot \hat{\mathbf{n}}_i ds$ out of R through C_1, C_2 and C_3, more than, less than or equal to the flux out of R through C? Show your reasoning.

26. The equations

$$z = 12, \quad x^2 + y^2 \le 25$$

describe a disk S of radius 5 lying in the plane $z = 12$. Compute the flux of $\vec{F} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\imath}} + z\hat{\mathbf{k}}$ across S.

27. Suppose a temperature function is given in space by $T(x, y, z) = x^2 + y^2 + z^2$ and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ with the outward normal. Find the heat flux across the surface S.

28. Compute the scalar curl and the divergence of the following plane vector fields.

    (a) $\vec{F} = 3xy\hat{\mathbf{i}} - \cos(x - y)\hat{\mathbf{j}}$

    (b) $\vec{F} = e^{x+2y}\hat{\mathbf{i}} + \arctan(xy)\hat{\mathbf{j}}$

29. Compute the vector curl $\nabla \times \vec{F}$ and the divergence of the following vector fields.

    (a) $\vec{F} = x\hat{\mathbf{i}} + e^{xy}\hat{\mathbf{k}}$

    (b) $\vec{F} = e^{x+2y}\hat{\mathbf{i}} + \arctan(xy)\hat{\mathbf{j}}$

    (c) $\vec{F} = \sqrt{1 + x^2}\hat{\mathbf{j}} - 2xyz\hat{\mathbf{i}} + \tan z\hat{\mathbf{k}}$

30. Let S be the surface formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane and let $\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 2(1 - z)\hat{\mathbf{k}}$. Calculate the flux of $\vec{F}$ across S, taking the upward direction as the one for which the flux is positive. Do this in two ways:

    (a) by direct calculation of $\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS$;

    (b) by computing the flux of $\vec{F}$ across a simpler surface and using the divergence theorem.

31. Use the divergence theorem to compute the flux of $\vec{F} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ outwards across the closed surface $x^4 + y^4 + z^4 = 1$.

32. Let $\vec{F} = -2xz\hat{\mathbf{i}} + y^2\hat{\mathbf{k}}$.

    (a) Calculate $\nabla \times \vec{F}$.

    (b) Show that $\iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = 0$ for any portion R of the unit sphere $x^2 + y^2 + z^2 = 1$. (take the normal vector fi pointing outward)

    (c) Show that $\int_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

33. Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying in $z > 1$. Orient S upwards and give its bounding circle, C, lying in $z = 1$ the compatible orientation.

    (a) Parametrize C and use the parametrization to evaluate the line integral

    $$I = \int_C xz dx + y dy + y dz.$$

    (b) Compute the vector curl of the vector field $\vec{F} = xz\hat{\mathbf{i}} + y\hat{\mathbf{j}} + y\hat{\mathbf{k}}$.

    (c) Write down a flux integral through S which can be computed using the value of I.

34. (a) Let C be a simple closed plane curve going counterclockwise around a region R. Let $M = M(x, y)$. Express $\int_C M dx$ as a double integral over R.

    (b) The mass of a plane object R with density function $\delta(x, y)$ is given by $\iint_R \delta(x, y) dA$. Find $M(x, y)$ so that $\int_C M dx$ is the mass of R with density function $\delta(x, y) = (x + y)^2$.

35. Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of $\vec{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ upward through S by reducing the surface integral to a double integral over the disk $x^2 + y^2 < 1$.

36. Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 4$ lying in $x^2 + y^2 > 1$. This is to say, lying outside the cylinder of radius 1 with axis the z-axis.

(a) Compute the flux outward through S of $\vec{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

(b) Show that the flux of $\vec{F}$ through any part of the cylindrical surface is zero.

(c) Using the divergence theorem applied to $\vec{F}$, compute the volume of the region between S and the cylinder.

37. (a) Show that the vector field

$$\vec{F} = \left(e^x yz, e^x z + 2yz, e^x y + y^2 + 1\right)$$

is conservative.

(b) By integrating on a curve find a potential for $\vec{F}$.

(c) Show that the vector field $\vec{G} = (y, x, y)$ is not conservative.