MATH 20E – Final Exam: solutions to practice problems

Alina Bucur

Problem 2 (a) $\nabla f = (y - 4x^3, x)$, so at $(1,1)$ it becomes $(3,1)$.

(b) $\Delta w \approx -3\Delta x + \Delta y$.

Problem 3 $\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y f_u + \dfrac{1}{y} f_v$ and $\dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x f_u - \dfrac{x}{y^2} f_v$.

Problem 4 (a) $\nabla f = (2xy^2 - 1, 2x^2 y)$, so at $(2,1)$ it becomes $(3,8)$.

(b) $z - 2 = 3(x-2) + 8(y-1)$ or $3x + 8y - z = 12$.

(c) $\Delta x = 1.9 - 2 = -0.1$ and $\Delta y = 1.1 - 1 = 0.1$; so $f(1.9, 1.1) - f(2,1) \approx 3\Delta x + 8\Delta y = -0.3 + 0.8 = 0.5$; since $f(2,1) = 2$, we obtain $f(1.9, 1.1) \approx 2.5$

Problem 5 (a) $w_x = w_u u_x + w_v v_x = -\dfrac{y}{x^2} w_u + 2x w_v$ and $w_y = w_u u_y + w_v v_y = \dfrac{1}{x} w_u + 2y w_v$.

(b) $x w_x + y w_y = x\left(-\dfrac{y}{x^2} w_u + 2x w_v\right) + y\left(\dfrac{1}{x} w_u + 2y w_v\right) = \left(-\dfrac{y}{x} + \dfrac{y}{x}\right) w_u + (2x^2 + 2y^2) w_v = 2v w_v$

(c) $x w_x + y w_y = 2v w_v = 2v(5v^4) = 10v^5$.

Problem 6 $\mathrm{vol}(R) = \displaystyle\iiint_R dV$

The equation of the sphere is $x^2 + y^2 + (z-2)^2 = 16$.
The shadow of R on the xy-plane is given by the quarter of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that sits in the first quadrant $x, y \geq 0$. So $0 \leq x \leq 2$ and for each x we have $0 \leq y \leq 3\sqrt{1 - \frac{x^2}{4}}$. For each point (x,y) in the shadow of R, have $0 \leq z \leq 2 + \sqrt{16 - x^2 - y^2}$. Hence

$$\mathrm{vol}(R) = \int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} \int_0^{2+\sqrt{16-x^2-y^2}} dz\, dy\, dx.$$

Problem 7 The two surfaces are paraboloids. The shadow of the region on the xy-plane is determined by the intersection of these two paraboloids. In other words, we need $z = 4 - x^2 - y^2$ to sit underneath $z = 10 - 4x^2 - 4y^2$, i.e. $4 - x^2 - y^2 \leq 10 - 4x^2 - 4y^2$. That is, we need $3x^2 + 3y^2 \leq 6 \Leftrightarrow x^2 + y^2 \leq 2$. So,

$$\mathrm{vol} = \iint_{x^2+y^2\leq 2} \int_{4-x^2-y^2}^{10-4x^2-4y^2} dV.$$

From here it's best to switch to cylindrical coordinates, so

$$\text{vol} = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{4-r^2}^{10-4r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r(6-3r^2)\,dr\,d\theta = \int_0^{2\pi}\left[3r^2 - \frac{3}{4}r^4\right]_{r=0}^{r=\sqrt{2}} d\theta = 6\pi.$$

Problem 8 The region R is the triangle formed by the lines $y = x\sqrt{3}$, $y = x$ and $x = 2$.
The angle made by the line $y = x\sqrt{3}$ with the positive x-axis is $\pi/3$, while the angle made by the line $y = x$ with the positive x-axis is $\pi/4$. The line $x = 2$ crosses the two lines at $(2, 2\sqrt{3})$ and $(2, 2)$. The line $x = 2$ is given in polar coordinates by $r\cos\theta = 2$, hence $r = \frac{2}{\cos\theta}$.

$$\int_0^2\int_x^{x\sqrt{3}} x\,dy\,dx = \int_{\pi/4}^{\pi/3}\int_0^{2/\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{8}{3}\int_{\pi/4}^{\pi/3}\frac{1}{\cos^2\theta} = \frac{8}{3}\Big[\tan\theta\Big]_{\theta=\pi/4}^{\theta=\pi/3} = \frac{8}{3}(\sqrt{3}-1).$$

Problem 9 (a) The region of integration is the triangle made by the lines $y = x$, $y = 2x$ and $x = 1$. It has vertices $(0, 0)$, $(1, 1)$ and $(1, 2)$.

(b) For $0 \le y \le 1$, have $y/2 \le x \le y$ and for $1 \le y \le 2$, have $y/2 \le x \le 1$. So

$$\int_0^1\int_x^{2x} dydx = \int_0^1\int_{y/2}^{y} dxdy + \int_1^2\int_{y/2}^{1} dxdy.$$

Problem 10 The ball of radius 5 is the solid B given by $x^2 + y^2 + z^2 \le 25$. Its shadow on the xy-plane is the disk of radius 5. In rectangular coordinates

$$\text{vol}(B) = \iiint_B dV = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} dzdydx.$$

Using the Jacobian of the spherical coordinate change, we have $dzdydx = \rho^2\sin\phi d\rho d\phi d\theta$. The ball of radius 5 has equation $\rho \le 5$. Thus θ takes any value in $[0, 2\pi)$ and ϕ takes any value in $[0, \pi]$ and we have

$$\text{vol}(B) = \int_0^{2\pi}\int_0^{\pi}\int_0^5 \rho^2\sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=0}^{\rho=5} d\phi d\theta = \frac{5^3}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\phi=0}^{\phi=\pi} d\theta$$

This equals

$$2\cdot\frac{5^3}{3}(2\pi) = \frac{500\pi}{3}.$$

Problem 11 The region is inside the the unit circle $x^2+y^2 = 1$, outside $x^2+y^2 = 2y \iff x^2+(y-1)^2 = 1$ the circle of radius 1 centered at $(0, 1)$ and with $x, y \ge 0$. In the uv-plane this becomes the triangle T with sides $u = 1, v = 0$ and $u = v$. Its vertices are at $(0, 0), (1, 0)$ and $(1, 1)$. We need to compute the Jacobian of the transformation

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & 2y-2 \end{vmatrix} = -4x.$$

We have to take absolute value, and since $x \ge 0$, we get $dudv = 4xdxdy$. Therefore

$$\int_0^{1/2}\int_{\sqrt{2y-y^2}}^{\sqrt{1-y^2}} xe^y dxdy = \frac{1}{4}\iint_T e^y dudv = \frac{1}{4}\int_0^1\int_0^u e^{u/2-v/2}dvdu = \frac{1}{4}\int_0^1\left[-2e^{u/2-v/2}\right]_{v=0}^{v=u} du$$

$$= \frac{1}{4}\int_0^1(2e^{u/2}-2)du = \left[e^{u/2} - \frac{u}{2}\right]_{u=0}^{u=1} = \sqrt{e} - \frac{3}{2}.$$

Problem 12 (a) $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = \dfrac{3x^2}{y}$.
Therefore
$$dudv = \frac{3x^2}{y}dxdy = 3udxdy \implies dxdy = \frac{1}{3u}dudv.$$

(b) $\displaystyle\int_1^5 \int_2^4 \frac{1}{3u} dvdu = \frac{2}{3}\int_1^5 \frac{1}{u}du = \frac{2}{3}\ln 5.$

Problem 14 Need to check that $\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$.

(a) yes: $\vec{F}(\vec{c}(t)) = \vec{F}(t^3, \sqrt{t}, -t, -\log t) = \left(3t^2, \dfrac{1}{2\sqrt{t}}, \dfrac{t^2}{-t^3}, -\dfrac{1}{t}\right) = \vec{c}\,'(t)$.

(b) yes: $\vec{F}(\vec{c}(t)) = (-\sin t, \cos t) = \vec{c}\,'(t)$.

(c) no: $\vec{F}(\vec{c}(t)) = (-\cos t, \sin t) \neq \vec{c}\,'(t) = (\cos t, -\sin t)$.

(d) no: $\vec{F}(x,y) = (-\sin t, t) \neq \vec{c}\,'(t) = (1, \cos t)$.

Problem 15 Want to have $\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$, i.e. $(e^{2t}+2te^{2t}, b, 2e^{2t}) = (ate^{at}+e^{at}, 1, 2e^{2t})$. Hence $a = 2, b = 1$.

Problem 16 Need to see if $\operatorname{curl}\vec{F} = N_x - M_y$ is 0.

(a) $M = x, N = -y$ so $M_y = 0, N_x = 0$. Since they are equal, this is a gradient field. Potential is $f(x,y)$ such that $\vec{F} = \nabla f$, i.e. $f_x = x, f_y = -y$. Therefore $f(x,y) = \frac{x^2}{2} + c(y)$ and $c'(y) = -y$. So $f(x,y) = \frac{x^2-y^2}{2} + \text{const}$.

(b) $M = y, N = y^2$ so $M_y = 1, N_x = 0$. This is not a gradient field.

(c) $M = 2xy, N = x^2+y^2$ so $M_y = 2x, N_x = 2x$. Hence $\vec{F}$ is a gradient field. Want to find $f(x,y)$ such that $\vec{F} = \nabla f$, i.e. $f_x = 2xy, f_y = x^2+y^2$. From the first relation we get $f(x,y) = x^2y + c(y)$. Plugging into second, get $x^2 + c'(y) = x^2 + y^2$ so $c(y) = y^3/3 + \text{const}$. Thus $f(x,y) = x^2y + \frac{y^3}{3} + \text{const}$.

Problem 17 $\vec{F} = (ax^2y + y^3 + 1)\hat{\mathbf{\i}} + (2x^3 + bxy^2 + 2)\hat{\mathbf{\j}}$

(a) Want $\frac{\partial}{\partial y}(ax^2y + y^3 + 1) = \frac{\partial}{\partial x}(2x^3 + bxy^2 + 2)$, i.e. $ax^2 + 3y^2 = 6x^2 + by^2$. Thus $a = 6, b = 3$.

(b) $\vec{F} = (6x^2y + y^3 + 1)\hat{\mathbf{\i}} + (2x^3 + 3xy^2 + 2)\hat{\mathbf{\j}}$.
We will integrate on the line segments from $(0,0)$ to $(x_1, 0)$ and then to (x_1, y_1). On the first segment: $x = t, y = 0, 0 \le t \le x_1, dx = dt, dy = 0$ so we get $\int_0^{x_1} 1dt = x_1$. On the second segment: $x = x_1, y = t, 0 \le t \le y_1, dx = 0, dy = dt$ so we get $\int_0^{y_1}(2x_1^3 + 3x_1t^2 + 2)dt = 2x_1^3y_1 + x_1y_1^3 + 2y_1$. Adding them up we get $2x_1^3y_1 + x_1y_1^3 + 2y_1 + x_1$, so the potential is
$$f(x,y) = 2x^3y + xy^3 + x + 2y.$$
Check: $\nabla f = (6x^2y + y^3 + 1, 2x^3 + 3xy^2 + 2) = \vec{F}$.

(c) C starts at $(1,0)$ and ends at $(-e^{\pi}, 0)$, so FTC tells us that
$$\int_C \vec{F}\cdot d\vec{r} = f(-e^{\pi}, 0) - f(1,0) = -e^{-\pi} - 1.$$

Problem 18 (a) $N_x = -12y = M_y$, hence $\vec{F}$ is conservative.

(b) $f_x = 3x^2 - 6y^2 \implies f = x^3 - 6xy^2 + c(y) \implies f_y = -12xy + c'(y) = -12xy + 4y$. So $c'(y) = 4y \implies c(y) = 2y^2 +$ const. In conclusion,
$$f = x^3 - 6xy^2 + 2y^2 (+\text{ constant}).$$

(c) C starts at $(1,0)$ and ends at $(1,1)$, so

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(1,0) = (1-6+2) - 1 = -4.$$

Problem 19 $\int_C yx^3 dx + y^2 dy = \int_0^1 x^2 x^3 dx + \left(x^2\right)^2 (2x dx) = \int_0^1 3x^5 dx = \frac{1}{2}.$

Problem 20 (a) The parametrization of the unit circle C is $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$. Then $dx = -\sin t dt, dy = \cos t dt$ and

$$\text{work} = \int_C M dx + N dy = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t dt) + (1 + \cos(\sin t))(\cos t dt).$$

Hence

$$\text{work} = \int_0^{2\pi} -(5\cos t + 3\sin t)\sin t + (1+\cos(\sin t))\cos t \, dt.$$

(b) Let R be the unit disk inside C. By Green's theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y) dA = \iint_R (0-3) dA = -3\text{area}(R) = -3\pi.$$

Problem 21 (a) Since we rotate around the y-axis, we need to write x as a function of y. Have $x = y^2, 0 \leq y \leq 1$. The area of the surface of revolution is then

$$\int_0^1 |y|\sqrt{1+(2y)^2} dy = \int_0^1 y\sqrt{1+(2y)^2} dy.$$

Taking $u = 1 + 4y^2$, we have $du = 8y dy$ so

$$\int_0^1 y\sqrt{1+4y^2} dy = \int_1^5 \sqrt{u}\frac{du}{8} = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=5} = \frac{1}{12}(5\sqrt{5}-1).$$

(b) Three of the faces of the tetrahedron are right triangles with sides equal to 1 and hypothenuse $\sqrt{2}$. The are of each of them is $1/2$, so their total area is $3/2$. The fourth face is an equilateral triangle with all sides equal to $\sqrt{2}$. The area of such a triangle is given by

$$\frac{\sqrt{2}\sqrt{2}\sin\pi/3}{2} = \frac{\sqrt{3}}{2}.$$

The total surface area is therefore $3/2 + \sqrt{3}/2$.

(c) area $= \iint_S dS = \iint_{u^2+v^2 \leq 4} \|(x_u, y_u, z_u) \times (x_v, y_v, z_v)\| du dv.$

$$(x_u, y_u, z_u) \times (x_v, y_v, z_v) = (1, 1, v) \times (-1, 1, u) = (u - v, -u - v, 2)$$

and its length is $\sqrt{(u-v)^2 + (u+v)^2 + 4} = \sqrt{2u^2 + 2v^2 + 4}$. The are is therefore

$$\iint_{u^2+v^2\leq 4} \sqrt{2(u^2+v^2)+4} = \int_0^2 \int_0^{2\pi} \sqrt{2r^2+4} r d\theta dr = 2\pi \int_0^2 \sqrt{2r^2+4} r dr$$

by switching to polar coordinates. Next we can set $w = 2r^2 + 4$ and get

$$2\pi \int_4^{12} w^{1/2}\frac{dw}{4} = \frac{\pi}{3}\left[w^{3/2}\right]_{w=4}^{w=12} = \frac{8\pi}{3}(3\sqrt{3}-1).$$

Problem 22 see book

Problem 23 (a) normal vector is the gradient, i.e. $(2x - y^6\sin(xy), 5y^4\cos(xy) - xy^5\sin(xy), -6z^5)$. At $(9, 0, 3)$ it becomes $(18, 0, -1458)$. The tangent plane is therefore

$$18(x-9) - 1458(z-3) = 0.$$

(b) $f(-1, 1) = e^2$ and $f_x = 4xye^{2x^2y} + \cos(x+y), f_y = 2x^2e^{2x^2y} + \cos(x+y)$. At $x = -1, y = 1$ the partial derivatives are $f_x = -4e^2 + 1, f_y = 2e^2 + 1$. The tangent plane is therefore

$$z - e^2 = (1 - 4e^2)(x+1) + (2e^2+1)(y-1).$$

(c) The cylinder S is parametrized by $x = 5\cos u, y = v, z = 5\sin u$. A normal vector is $(x_u, y_u, z_u) \times (x_v, y_v, z_v) = (-5\sin u, 0, 5\cos u) \times (0, 1, 0)$. At $P = (3, 19, 4)$ we have $v = 19, \sin u = 4/5, \cos u = 3/5$. So the normal vector becomes $(-4, 0, 3) \times (0, 1, 0) = -3\hat{\mathbf{i}} - 4\hat{\mathbf{k}}$. The tangent plane is

$$-3(x-3) - 4(z-4) = 0 \iff 3x + 4z = 25.$$

Problem 24 (a) This is similar to Example 2, page 524.

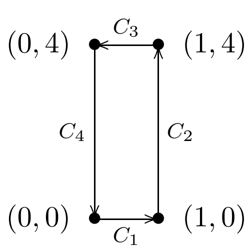
$$\text{area} = \frac{1}{2}\int_C x\,dy - y\,dx$$

where C is the curve $x^{2/5} + y^{2/5} = 32^{2/5}$ oriented counterclockwise. We parametrize $C : x = 32\cos^5 t y = 32\sin^5 t, 0 \le t \le 2\pi$. Then $dx = -160\cos^4 t \sin t dt$ and $dy = 160\sin^4 t \cos t dt$. Thus

$$\begin{aligned}
\text{area} = \frac{1}{2}\int_0^{2\pi} 160 \cdot 32(\sin^4 t\cos^6 t + \sin^6 t\cos^4 t)dt &= 2560\int_0^{2\pi} \sin^4 t\cos^4 t dt \\
= 160\int_0^{2\pi} \sin^4(2t)dt = 10\int_0^{2\pi}(1 - \cos(4t))^2 dt &= 10\int_0^{2\pi} 1 + \cos^2(4t) - 2\cos(4t)dt \\
&= 20\pi + 5\int_0^{2\pi}(1 - \sin 8t)dt - [10\sin(4t)]_{t=0}^{t=2\pi} = 30\pi.
\end{aligned}$$

(b) This is problem 19, page 530.

Problem 25 (a) $\hat{\mathbf{n}}_1 = -\hat{\mathbf{j}}, \hat{\mathbf{n}}_2 = \hat{\mathbf{i}}, \hat{\mathbf{n}}_3 = \hat{\mathbf{j}}, \hat{\mathbf{n}}_4 = -\hat{\mathbf{i}}$



(b) The flux out of R is the flux across $C = C_1 + C_2 + C_3 + C_4$ and is given by

$$\sum_{i=1}^{4}\int_{C_i} \vec{F} \cdot \hat{\mathbf{n}}_i ds = \iint_R \operatorname{div} \vec{F} dA$$

and $\operatorname{div} \vec{F} = y + \cos x \cos y - \cos x \cos y = y$. So

$$\iint_R \operatorname{div} \vec{F} dA = \int_0^4 \int_0^1 y dx dy = \int_0^4 y dy = 8.$$

(c) On C_4, $x = 0$, so $\vec{F} = -\sin y \hat{\mathbf{j}}$, whereas $\hat{\mathbf{n}}_4 = -\hat{\mathbf{\imath}}$. Hence $\vec{F} \perp \hat{\mathbf{n}}_4$ and $\vec{F} \cdot \hat{\mathbf{n}}_4 = 0$. Therefore the flux of $\vec{F}$ through C_4 equals 0. Thus

$$\sum_{i=1}^{3} \int_{C_i} = \sum_{i=1}^{4} \int_{C_i} \vec{F} \cdot \hat{\mathbf{n}}_i ds = \text{total flux out of } R.$$

Problem 26 This is Example 6, page 495, in the textbook.

Problem 27 This is Example 4, page 492, in the textbook.

Problem 28 (a) $\operatorname{curl} \vec{F} = N_x - M_y = \sin(x-y) - 3x, \operatorname{div} \vec{F} = M_x + N_y = 3y + \sin(x-y)$.

(b) $\operatorname{curl} \vec{F} = N_x - M_y = \dfrac{y}{1+x^2y^2} - 2e^{x+2y}, \operatorname{div} \vec{F} = M_x + N_y = e^{x+2y} + \dfrac{x}{1+x^2y^2}$.

Problem 29 (a) $\nabla \times \vec{F} = (0, -ye^{xy}, 0), \operatorname{div} F = P_x + Q_y + R_z = 1$.

(b) $\nabla \times \vec{F} = \left(0, 0, \dfrac{y}{1+x^2y^2} - 2e^{x+2y}\right), \operatorname{div} F = P_x + Q_y + R_z = e^{x+2y} + \dfrac{x}{1+x^2y^2}$.

(c) $\nabla \times \vec{F} = \left(0, -2xy, \dfrac{x}{\sqrt{1+x^2}} + 2xz\right), \operatorname{div} F = P_x + Q_y + R_z = -2yz + \dfrac{1}{\cos^2 z}$.

Problem 30 (a) S is the graph of $z = f(x,y) = 1 - x^2 - y^2$, and the normal points upwards, so $\hat{\mathbf{n}} dS = (-f_x, -f_y, 1) dA = (2x, 2y, 1) dA$.
Therefore

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_{\text{shadow}} (x, y, 2(1-z)) \cdot (2x, 2y, 1) dA = \iint_{\text{shadow}} 2x^2 + 2y^2 + 2(1-z) dA = \iint_{\text{shadow}} 4x^2 + 4y^2 dA$$

since $z = 1 - x^2 - y^2 \implies 1 - z = x^2 + y^2$.
Shadow = unit disk $x^2 + y^2 \le 1$; switching to polar coordinates, we have

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \int_0^{2\pi} \int_0^1 4r^2 r dr d\theta = 2\pi.$$

(b) Let R = unit disk in the xy-plane, with normal vector pointing down ($\hat{\mathbf{n}} = -\hat{\mathbf{k}}$). Then

$$\iint_R \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_R (x, y, 2) \cdot (-\hat{\mathbf{k}}) dS = -\iint_R -2 dS = -2\text{area}(R) = -2\pi.$$

By the divergence theorem,

$$\iint_{S+R} \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_W (\operatorname{div} \vec{F}) dV = 0,$$

since $\operatorname{div} \vec{F} = 1 + 1 - 2 = 0$. Therefore $\iint_S = -\iint_R = 2\pi$.

Problem 31 $\operatorname{div} \vec{F} = 0$, and so

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iiint_W \operatorname{div} \vec{F} dV = 0.$$

Problem 32 (a) $\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2xz & 0 & y^2 \end{vmatrix} = (2y, -2x, 0).$

(b) On the unit sphere, $\hat{\mathbf{n}} = (x, y, z)$ so $(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} = 2yx - 2xy = 0$. Therefore $\iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = 0$.

(c) By Stokes' Theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS$ where R is the region delimited by C on the unit sphere. Using the result of (b), we get $\int_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = 0$.

Problem 33 (a) $z = 1$ and $x^2 + y^2 + z^2 = 1$, so $x^2 + y^2 = 1$. Therefore C is the circle of radius 1 in the $z = 1$ plane. Compatible orientation: counterclockwise.
Parametrization: $x = \cos t, y = \sin t, z = 1$ Therefore $dx = -\sin t dt, dy = \cos t dt, dz = 0$.

$$I = \int_C xzdx + ydy + ydz = \int_0^{2\pi} (-\cos t \sin t + \cos t \sin t) dt = 0.$$

(b)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & y & y \end{vmatrix} = \hat{\mathbf{i}} + x\hat{\mathbf{j}}.$$

(c) By Stokes' Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS.$$

$\hat{\mathbf{n}}$ is the normal pointing upwards, so $\hat{\mathbf{n}} = \frac{(x,y,z)}{\sqrt{2}}$ on the upper hemisphere of radius $\sqrt{2}$. Thus

$$I = \int_C \vec{F} \cdot d\vec{r} = \iint_S (1, x, 0) \cdot \frac{(x, y, z)}{\sqrt{2}} dS = \iint_S \frac{x + xy}{\sqrt{2}} dS.$$

Problem 34 (a) By taking $N = 0$ in Green's theorem, we get $\int_C M dx = \iint_R -M_y dA$.

(b) We want $M(x, y)$ such that $-M_y = (x + y)^2$. Use $M = -\frac{1}{3}(x + y)^3$.

Problem 35 The surface is the graph of the function $f(x, y) = xy$ with x, y in the unit disk. The flux is upward, so

$$\hat{\mathbf{n}} dS = +(-f_x, -f_y, 1) dxdy = (-y, -x, 1) dxdy.$$

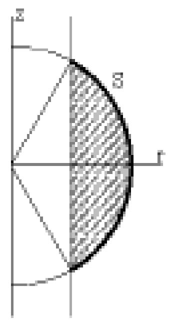
Hence

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_{x^2+y^2<1} (y, x, z) \cdot (-y, -x, 1) dxdy = \iint_{x^2+y^2<1} (-y^2 - x^2 + xy) dxdy$$

where we substituted $z = xy$. Using polar coordinates we get

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \int_0^{2\pi} \int_0^1 (-r^2 + r^2 \cos\theta \sin\theta) r dr d\theta.$$

- inner integral: $\displaystyle\int_0^1 (-r^2 + r^2 \cos\theta \sin\theta) r dr = \frac{1}{4}(\cos\theta \sin\theta - 1).$
- outer integral: $\displaystyle\int_0^{2\pi} \frac{1}{4}(\cos\theta \sin\theta - 1) d\theta = \frac{1}{4}\left[\frac{\sin^2\theta}{2} - \theta\right]_{\theta=0}^{\theta=2\pi} = -\frac{\pi}{2}.$



Problem 36 (a) Consider the figure

We have $\hat{\mathbf{n}} = \frac{1}{2}(x,y,z)$, hence $\vec{F} \cdot \hat{\mathbf{n}} = (y,-x,z) \cdot \frac{(x,y,z)}{2} = \frac{z^2}{2}$. For the part of the sphere of radius 2 we use the parametrization $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta, z = 2\cos\phi$. Then $dS = 2^2\sin\phi d\phi d\theta = 4sin\phi d\phi d\theta$.
Since we are outside the cylinder, the bounds are $0 \le \theta \le 2\pi$ and $\pi/6 \le \phi \le 5\pi/6$. (see figure) Thus the flux is given by

$$\int_0^{2\pi}\int_{\pi/6}^{5\pi/6} \frac{4\cos^2\phi}{2} 4\sin\phi d\phi d\theta = 8\int_0^{2\pi}\int_{\pi/6}^{5\pi/6} \cos^2\phi\sin\phi d\phi d\theta.$$

- inner integral: $\displaystyle\int_{\pi/6}^{5\pi/6} \cos^2\phi\sin\phi d\phi = \left[-\frac{\cos^3\phi}{3}\right]_{\phi=\pi/6}^{\phi=5\pi/6} = \frac{\sqrt{3}}{4}$.
- outer integral: $\displaystyle 8\int_0^{2\pi} \frac{\sqrt{3}}{4} d\theta = 4\pi\sqrt{3}$.

(b) On the cylinder $\hat{\mathbf{n}} = \pm(x,y,0)$ and so $\vec{F} \cdot \hat{\mathbf{n}} = 0$. Therefore the flux is 0.

(c) $\operatorname{div} \vec{F} = 1$, hence

$$\operatorname{vol}(W) = \iiint_W 1dV = \iiint_W (\operatorname{div}\vec{F})dV = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS + \iint_{\text{cylinder}} \vec{F} \cdot \hat{\mathbf{n}} dS = 4\pi\sqrt{3} + 0 = 4\pi\sqrt{3}.$$

Problem 37 (a)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x yz & e^x z + 2yz & e^x y + y^2 + 1 \end{vmatrix} = (e^x + 2y - e^x - 2y)\hat{\mathbf{i}} - (e^x y - e^x y)\hat{\mathbf{j}} + (e^x z - e^x z)\hat{\mathbf{k}} = 0.$$

(b) On the segment from $(0,0,0)$ to $(x_1,0,0) : x = t, y = 0, z = 0, 0 \le t \le x_1, dx = dt, dy = 0 = dz$ so we get $\int_0^{x_1} 0dt = 0$.
On the segment from $(x_1,0,0)$ to $(x_1,y_1,0) : x = x_1, y = t, z = 0, 0 \le t \le y_1, dx = 0, dy = dt, dz = 0$ so we get $\int_0^{y_1} 0dt = 0$.
On the segment from $(x_1,y_1,0)$ to $(x_1,y_1,z_1) : x = x_1, y = y_1, z = t, 0 \le t \le z_1, dx = 0 = dy, dz = dt$ so we get $\int_0^{z_1}(e^{x_1}y + y_1^2 + 1)dt = (e^{x_1}y + y_1^2 + 1)z_1$.
Therefore

$$f(x,y,z) = e^x yz + y^2 z + z.$$

Check: $\nabla f = (f_x, f_y, f_z) = (e^x yz, e^x z + 2yz, e^x y + y^2 + 1) = \vec{F}$.

(c)

$$\nabla \times \vec{G} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & y \end{vmatrix} = \hat{\mathbf{i}} \neq 0.$$