MATH 20E – Midterm 1 Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the online notes, the relevant sections of the book and the homework problems. If you still have trouble with some of the topics encountered so far, take the book (or another calculus book) and solve more problems related to that topic until you *really* understand how and why things work. It is especially important that you are familiar with the concepts from MATH 20C, even though the problems below do not directly address all those topics.

Topics from MATH 20C

vectors; operation with vectors (addition, substraction, multiplication by scalars, dot product, cross product); determinants, area, volume
calculus with vectors: limits, differentiation (product rules, chain rule), integration
functions of several variables: graphs, level curves and surfaces, contour maps; limits, partial derivatives, gradient, chain rule
double integrals (rectangular and polar coordinates); triple integrals (rectangular and cylindrical coordinates); pay special attention to how you setup the integration

Topics from MATH 20E

functions of several variables, partial derivatives, chain rule, approximation formula, tangent planes; derivative matrix;
integration in several variables, change of variables
vector fields

Concept review

1. Make sure you know how to draw the level curves and the contour map of a function.
2. What are the first order partial derivatives of a function of 2 variables? What about a function of 3 variables? What do they represent geometrically? How do you compute them?
3. Chain rule (all its forms).
4. What are the higher order partial derivatives of a function?
5. Compute $\frac{\partial^3 f}{\partial x \partial z^2}$ and $\frac{\partial^3 f}{\partial x \partial y \partial z}$ of $f(x, y, z) = xe^{-yz}$.
6. What is the gradient of a function?
7. Compute the gradient of $f(x, y, z) = xe^{-y^2 z}$.
8. Write down the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0)$.

9. What does a double/triple integral represent geometrically?
10. More double/triple integrals: make sure you can draw the picture of the region, take slices to set up the iterated integral. And make sure you can interchange the order of integration.
11. Make sure you can set up double integrals in polar coordinates; recall that $dA = rdrd\theta$.
12. Make sure you can set up triple integrals in cylindrical coordinates; recall that $dV = rdzdrd\theta$.
13. Make sure you can set up triple integrals in spherical coordinates; recall that $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.
14. Make sure you can compute the Jacobian of a general change of variables.
15. Make sure you understand how to set up an integral after a change of variables (bounds and integrand).
16. What is a vector field in $\mathbb{R}^n$?
17. Make sure you can give a couple of examples from the real world.
18. Make sure you can sketch a vector field in the plane.
19. What is a flow line?
20. Is $\vec{c}(t) = (t, \sin t)$ a flow line for $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$?
21. What is a gradient field? What is its potential?
22. How do you test to see if a vector field is a gradient field?
23. How do you find the potential of a gradient field?
24. Find the potential of the vector field $\vec{F} = (2x + y)\hat{\mathbf{i}} + (x + 2y)\hat{\mathbf{j}}$.

Practice problems

This is a collection of extra problems to help you prepare for the midterm. Once again, the list is not complete and these problems do **not** imply anything about the content of the exam.

1. Do HW3.
2. Go over HW1 and HW2. Unless you got everything right on both of those, do also HW0.
3. Let $f(x, y) = xy - x^4$.
   (a) Find the gradient of f at $(1, 1)$.
   (b) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.
4. Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.
5. Let $f(x, y) = x^2y^2 - x$.
   (a) Find ∇f at $(2, 1)$.
   (b) Write the equation for the tangent plane to the graph of f at $(2, 1, 2)$.
   (c) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.
6. Let $u = y/x$, $v = x^2 + y^2$, $w = w(u, v)$.

(a) Express the partial derivatives w_x and w_y in terms of x, y, w_u and w_v.

(b) Express $xw_x + yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.

(c) Find $xw_x + yw_y$ in case $w = v^5$.

7. Set up an iterated integral giving the volume of the solid in space in the shape of the region bounded by the elliptical cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the sphere of radius 4 centered at $(0, 0, 2)$ in the first octant $x, y, z \leq 0$. Give the integrand and bounds, but DO NOT EVALUATE.

8. Find the volume of the region defined by $4 - x^2 - y^2 \leq z \leq 10 - 4x^2 - 4y^2$.

9. Sketch the region of integration and evaluate by changing to polar coordinates

$$\int_0^2 \int_x^{x\sqrt{3}} x\, dy\, dx.$$

10. (a) Draw a picture of the region of integration of $\int_0^1 \int_x^{2x} dydx$.

(b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$. Warning: your answer will have two pieces.

11. Set up a triple integral to compute the volume of the ball of radius 5. Now evaluate the integral using spherical coordinates.

12. Evaluate

$$\int_0^{1/2} \int_{\sqrt{2y-y^2}}^{\sqrt{1-y^2}} xe^y dxdy$$

by changing variables to $u = x^2 + y^2$ and $v = x^2 + y^2 - 2y$. (First you should draw a picture of the region of integration.)

13. Sketch the following vector fields:

(a) $\vec{F}(x, y) = (x, -y)$

(b) $\vec{F}(x, y) = (y, y^2)$

14. In which of the following cases is $\vec{c}(t)$ a flow line for the vector field $\vec{F}$?

(a) $\vec{F}(x_1, x_2, x_3, x_4) = \left(3x_3^2, \frac{1}{2x_2}, \frac{x_1}{x_3^3}, \frac{1}{x_3}\right)$ $\quad \vec{c}(t) = (t^3, \sqrt{t}, -t, -\log t)$ for $t > 0$.

(b) $\vec{F}(x, y) = (-y, x)$ $\quad \vec{c}(t) = (\cos t, \sin t)$.

(c) $\vec{F}(x, y) = (-y, x)$ $\quad \vec{c}(t) = (\sin t, \cos t)$.

(d) $\vec{F}(x, y) = (-y, x)$ $\quad \vec{c}(t) = (t, \sin t)$.

15. Determine a, b such that $\vec{c}(t) = (te^{at}, t, e^{2t})$ is a flow line for the vector field $\vec{F} = (z + 2yz)\hat{\mathbf{i}} + b\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$.

16. Find the potential for the following vector fields or show that the potential does not exist.

(a) $\vec{F}(x, y) = (x, -y)$

(b) $\vec{F}(x, y) = (y, y^2)$

(c) $\vec{F}(x, y) = (2xy, x^2 + y^2)$

17. Let $\vec{F} = (ax^2y + y^3 + 1)\hat{\mathbf{i}} + (2x^3 + bxy^2 + 2)\hat{\mathbf{j}}$ be a vector field, where a and b are constants.

(a) Find the values of a and b for which $\vec{F}$ is a gradient field.

(b) For these values of a and b, find $f(x, y)$ such that $\vec{F} = \nabla f$.