MATH 20E – Midterm 1: solutions to practice problems

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Problem 3 (a) $\nabla f = (y - 4x^3, x)$, so at $(1,1)$ it becomes $(3,1)$.

(b) $\Delta w \approx -3\Delta x + \Delta y$.

Problem 4 $\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y f_u + \dfrac{1}{y} f_v$ and $\dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x f_u - \dfrac{x}{y^2} f_v$.

Problem 5 (a) $\nabla f = (2xy^2 - 1, 2x^2 y)$, so at $(2,1)$ it becomes $(3,8)$.

(b) $z - 2 = 3(x-2) + 8(y-1)$ or $3x + 8y - z = 12$.

(c) $\Delta x = 1.9 - 2 = -0.1$ and $\Delta y = 1.1 - 1 = 0.1$; so $f(1.9, 1.1) - f(2,1) \approx 3\Delta x + 8\Delta y = -0.3 + 0.8 = 0.5$; since $f(2,1) = 2$, we obtain $f(1.9, 1.1) \approx 2.5$

Problem 6 (a) $w_x = w_u u_x + w_v v_x = -\dfrac{y}{x^2} w_u + 2x w_v$ and $w_y = w_u u_y + w_v v_y = \dfrac{1}{x} w_u + 2y w_v$.

(b) $x w_x + y w_y = x\left(-\dfrac{y}{x^2} w_u + 2x w_v\right) + y\left(\dfrac{1}{x} w_u + 2y w_v\right) = \left(-\dfrac{y}{x} + \dfrac{y}{x}\right) w_u + (2x^2 + 2y^2) w_v = 2v w_v$

(c) $x w_x + y w_y = 2v w_v = 2v(5v^4) = 10v^5$.

Problem 7 $\mathrm{vol}(R) = \displaystyle\iiint_R dV$

The equation of the sphere is $x^2 + y^2 + (z-2)^2 = 16$.
The shadow of R on the xy-plane is given by the quarter of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that sits in the first quadrant $x, y \geq 0$. So $0 \leq x \leq 2$ and for each x we have $0 \leq y \leq 3\sqrt{1 - \frac{x^2}{4}}$. For each point (x,y) in the shadow of R, have $0 \leq z \leq 2 + \sqrt{16 - x^2 - y^2}$. Hence

$$\mathrm{vol}(R) = \int_0^2 \int_0^{3\sqrt{1-\frac{x^2}{4}}} \int_0^{2+\sqrt{16-x^2-y^2}} dz\, dy\, dx.$$

Problem 8 The two surfaces are paraboloids. The shadow of the region on the xy-plane is determined by the intersection of these two paraboloids. In other words, we need $z = 4 - x^2 - y^2$ to sit underneath $z = 10 - 4x^2 - 4y^2$, i.e. $4 - x^2 - y^2 \leq 10 - 4x^2 - 4y^2$. That is, we need $3x^2 + 3y^2 \leq 6 \Leftrightarrow x^2 + y^2 \leq 2$. So,

$$\mathrm{vol} = \iint_{x^2+y^2 \leq 2} \int_{4-x^2-y^2}^{10-4x^2-4y^2} dV.$$

From here it's best to switch to cylindrical coordinates, so

$$\text{vol} = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{4-r^2}^{10-4r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r(6-3r^2)\,dr\,d\theta = \int_0^{2\pi}\left[3r^2 - \frac{3}{4}r^4\right]_{r=0}^{r=\sqrt{2}} d\theta = 6\pi.$$

Problem 9 The region R is the triangle formed by the lines $y = x\sqrt{3}$, $y = x$ and $x = 2$.
The angle made by the line $y = x\sqrt{3}$ with the positive x-axis is $\pi/3$, while the angle made by the line $y = x$ with the positive x-axis is $\pi/4$. The line $x = 2$ crosses the two lines at $(2, 2\sqrt{3})$ and $(2, 2)$. The line $x = 2$ is given in polar coordinates by $r\cos\theta = 2$, hence $r = \frac{2}{\cos\theta}$.

$$\int_0^2\int_x^{x\sqrt{3}} x\,dy\,dx = \int_{\pi/4}^{\pi/3}\int_0^{2/\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{8}{3}\int_{\pi/4}^{\pi/3}\frac{1}{\cos^2\theta} = \frac{8}{3}\Big[\tan\theta\Big]_{\theta=\pi/4}^{\theta=\pi/3} = \frac{8}{3}(\sqrt{3}-1).$$

Problem 10 (a) The region of integration is the triangle made by the lines $y = x$, $y = 2x$ and $x = 1$. It has vertices $(0,0)$, $(1,1)$ and $(1,2)$.

(b) For $0 \le y \le 1$, have $y/2 \le x \le y$ and for $1 \le y \le 2$, have $y/2 \le x \le 1$. So

$$\int_0^1\int_x^{2x} dydx = \int_0^1\int_{y/2}^{y} dxdy + \int_1^2\int_{y/2}^{1} dxdy.$$

Problem 11 The ball of radius 5 is the solid B given by $x^2 + y^2 + z^2 \le 25$. Its shadow on the xy-plane is the disk of radius 5. In rectangular coordinates

$$\text{vol}(B) = \iiint_B dV = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} dzdydx.$$

Using the Jacobian of the spherical coordinate change, we have $dzdydx = \rho^2\sin\phi d\rho d\phi d\theta$. The ball of radius 5 has equation $\rho \le 5$. Thus θ takes any value in $[0, 2\pi)$ and ϕ takes any value in $[0, \pi]$ and we have

$$\text{vol}(B) = \int_0^{2\pi}\int_0^{\pi}\int_0^5 \rho^2\sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=0}^{\rho=5} d\phi d\theta = \frac{5^3}{3}\int_0^{2\pi}[-\cos\phi]_{\phi=0}^{\phi=\pi}\,d\theta$$

This equals

$$2\cdot\frac{5^3}{3}(2\pi) = \frac{500\pi}{3}.$$

Problem 12 The region is inside the the unit circle $x^2+y^2 = 1$, outside $x^2+y^2 = 2y \iff x^2+(y-1)^2 = 1$ the circle of radius 1 centered at $(0,1)$ and with $x, y \ge 0$. In the uv-plane this becomes the triangle T with sides $u = 1, v = 0$ and $u = v$. Its vertices are at $(0,0), (1,0)$ and $(1,1)$. We need to compute the Jacobian of the transformation

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & 2y-2 \end{vmatrix} = -4x.$$

We have to take absolute value, and since $x \ge 0$, we get $dudv = 4xdxdy$. Therefore

$$\int_0^{1/2}\int_{\sqrt{2y-y^2}}^{\sqrt{1-y^2}} xe^y dxdy = \iint_T e^y dudv = \int_0^1\int_0^u e^{u/2-v/2}dvdu = \int_0^1\left[-2e^{u/2-v/2}\right]_{v=0}^{v=u} du$$
$$= \int_0^1 (2e^{u/2}-2)du = \left[4e^{u/2} - 2u\right]_{u=0}^{u=1} = 4\sqrt{e} - 6.$$

Problem 13 omitted

Problem 14 Need to check that $\vec{F}(\vec{c}(t)) = \vec{c}'(t)$.

(a) yes: $\vec{F}(\vec{c}(t)) = \vec{F}(t^3, \sqrt{t}, -t, -\log t) = \left(3t^2, \dfrac{1}{2\sqrt{t}}, \dfrac{t^2}{-t^3}, -\dfrac{1}{t}\right) = \vec{c}'(t)$.

(b) yes: $\vec{F}(\vec{c}(t)) = (-\sin t, \cos t) = \vec{c}'(t)$.

(c) no: $\vec{F}(\vec{c}(t)) = (-\cos t, \sin t) \neq \vec{c}'(t) = (\cos t, -\sin t)$.

(d) no: $\vec{F}(x, y) = (-\sin t, t) \neq \vec{c}'(t) = (1, \cos t)$.

Problem 15 Want to have $\vec{F}(\vec{c}(t)) = \vec{c}'(t)$, i.e. $(e^{2t}+2te^{2t}, b, 2e^{2t}) = (ate^{at}+e^{at}, 1, 2e^{2t})$. Hence $a = 2, b = 1$.

Problem 16 Find the potential for the following vector fields or show that the potential does not exist.

(a) $M = x, N = -y$ so $M_y = 0, N_x = 0$. Since they are equal, this is a gradient field. Potential is $f(x, y)$ such that $\vec{F} = \nabla f$, i.e. $f_x = x, f_y = -y$. Therefore $f(x, y) = \frac{x^2}{2} + c(y)$ and $c'(y) = -y$. So $f(x, y) = \frac{x^2-y^2}{2} + \text{const}$.

(b) $M = y, N = y^2$ so $M_y = 1, N_x = 0$. This is not a gradient field.

(c) $M = 2xy, N = x^2 + y^2$ so $M_y = 2x, N_x = 2x$. Hence $\vec{F}$ is a gradient field. Want to find $f(x, y)$ such that $\vec{F} = \nabla f$, i.e. $f_x = 2xy, f_y = x^2 + y^2$. From the first relation we get $f(x, y) = x^2y + c(y)$. Plugging into second, get $x^2 + c'(y) = x^2 + y^2$ so $c(y) = y^3/3 + \text{const}$. Thus $f(x, y) = x^2y + \frac{y^3}{3} + \text{const}$.

Problem 17 Let $\vec{F} = (ax^2y + y^3 + 1)\mathbf{\hat{i}} + (2x^3 + bxy^2 + 2)\mathbf{\hat{j}}$ be a vector field, where a and b are constants.

(a) Want $\frac{\partial}{\partial y}(ax^2y + y^3 + 1) = \frac{\partial}{\partial x}(2x^3 + bxy^2 + 2)$, i.e. $ax^2 + 3y^2 = 6x^2 + by^2$. Thus $a = 6, b = 3$.

(b) $\vec{F} = (6x^2y + y^3 + 1)\mathbf{\hat{i}} + (2x^3 + 3xy^2 + 2)\mathbf{\hat{j}}$.
$f_x = 6x^2y + y^3 + 1 \implies f = 2x^3y + xy^3 + x + c(y)$. Therefore, $f_y = 2x^3 + 3xy^2 + c'(y)$. Setting this equal to N, we have $2x^3 + 3xy^2 + c'(y) = 2x^3 + 3xy^2 + 2$ so $c'(y) = 2$ and $c = 2y$. So

$$f = 2x^3y + xy^3 + x + 2y \ (+\text{constant}).$$