MATH 20E – Midterm 2 Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the online notes, the relevant sections of the book and the homework problems. If you still have trouble with some of the topics encountered so far, take the book (or another calculus book) and solve more problems related to that topic until you *really* understand how and why things work. The second midterm will focus on the material covered after the first midterm, but it builds on the material from the beginning of the term.

Topics

functions of several variables, partial derivatives, chain rule, approximation formula, tangent planes; derivative matrix;
integration in several variables, change of variables
surface area, tangent planes to surfaces
vector fields, line integrals, work, flux
Green's theorem
gradient and conservative vector fields
curl and div, Green's theorem (scalar form, vector form, divergence theorem in the plane)

Concept review

1. Chain rule (all its forms).
2. What is the gradient of a function?
3. Compute the gradient of $f(x, y, z) = xe^{-y^2 z}$.
4. Write down the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0)$.
5. Make sure you can set up double integrals in polar coordinates; recall that $dA = rdrd\theta$.
6. Make sure you can set up triple integrals in cylindrical coordinates; recall that $dV = rdzdrd\theta$.
7. Make sure you can set up triple integrals in spherical coordinates; recall that $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.
8. Make sure you can compute the Jacobian of a general change of variables.
9. Make sure you understand how to set up an integral after a change of variables (bounds and integrand).
10. What is a vector field in $\mathbb{R}^n$?
11. Make sure you can give a couple of examples from the real world.
12. Make sure you can sketch a vector field in the plane.
13. What is a flow line?

14. Is $\vec{c}(t) = (t, \sin t)$ a flow line for $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$?
15. What is a gradient field? What is its potential?
16. When is a vector field conservative?
17. How do you test to see if a vector field is a gradient field?
18. How do you find the potential of a gradient field?
19. Find the potential of the vector field $\vec{F} = (2x+y)\hat{\mathbf{i}} + (x+2y)\hat{\mathbf{j}}$.
20. What is the work done by a vector field along a trajectory?
21. Make sure you can parametrize curves (both in the plane and the space) and surfaces.
22. Make sure you can set up line and surface integrals, both for scalar functions and vector fields.
23. Make sure you can compute the tangent plane to a surface in the case of graphs, parametric surfaces, implicit surfaces.
24. Make sure you can find a normal vector to a surface in the case of graphs, parametric surfaces, implicit surfaces.
25. Make sure you can compute the surface area for graphs, parametric surfaces, implicit surfaces and surfaces of revolution.

Practice problems

This is a collection of extra problems to help you prepare for the midterm. Once again, the list is not complete and these problems do **not** imply anything about the content of the exam.

1. Go over HW.
2. Sketch the following vector fields:
   (a) $\vec{F}(x,y) = (x, -y)$
   (b) $\vec{F}(x,y) = (y, y^2)$
3. In which of the following cases is $\vec{c}(t)$ a flow line for the vector field $\vec{F}$?
   (a) $\vec{F}(x_1,x_2,x_3,x_4) = \left(3x_3^2, \frac{1}{2x_2}, \frac{x_1}{x_3^3}, \frac{1}{x_3}\right)$ $\quad \vec{c}(t) = (t^3, \sqrt{t}, -t, -\log t)$ for $t > 0$.
   (b) $\vec{F}(x,y) = (-y, x)$ $\quad \vec{c}(t) = (\cos t, \sin t)$.
   (c) $\vec{F}(x,y) = (-y, x)$ $\quad \vec{c}(t) = (\sin t, \cos t)$.
   (d) $\vec{F}(x,y) = (-y, x)$ $\quad \vec{c}(t) = (t, \sin t)$.
4. Determine a, b such that $\vec{c}(t) = (te^{at}, t, e^{2t})$ is a flow line for the vector field $\vec{F} = (z + 2yz)\hat{\mathbf{i}} + b\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$.
5. Find the potential for the following vector fields or show that the potential does not exist.
   (a) $\vec{F}(x,y) = (x, -y)$
   (b) $\vec{F}(x,y) = (y, y^2)$
   (c) $\vec{F}(x,y) = (2xy, x^2 + y^2)$

6. Let $\vec{F} = (ax^2y + y^3 + 1)\hat{\mathbf{i}} + (2x^3 + bxy^2 + 2)\hat{\mathbf{j}}$ be a vector field, where a and b are constants.

   (a) Find the values of a and b for which $\vec{F}$ is a gradient field.

   (b) For these values of a and b, find $f(x, y)$ such that $\vec{F} = \nabla f$.

   (c) Still using the values of a and b frompart(a), compute $\int_C \vec{F} \cdot d\vec{r}$ along the curve C such that $x = e^t \cos t, y = e^t \sin t, 0 \le t \le \pi$.

7. (a) Show that $\vec{F} = (3x^2 - 6y^2)\hat{\mathbf{i}} + (-12xy + 4y)\hat{\mathbf{j}}$ is conservative.

   (b) Find a potential function for $\vec{F}$.

   (c) Let C be the curve $x = 1 + y^3(1 - y)^3, 0 \le y \le 1$. Calculate $\int_C \vec{F} \cdot d\vec{r}$.

8. For $\vec{F} = yx^3\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$ find the work done by the $\vec{F}$ on the portion of the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$.

9. (a) Express the work done by the force field $\vec{F} = (5x + 3y)\hat{\mathbf{i}} + (1 + \cos y)\hat{\mathbf{j}}$ on a particle moving counterclockwise once around the unit circle centered at the origin in the form $\displaystyle\int_a^b f(t)dt$.(Do not evaluate the integral; dont even simplify $f(t)$.)

   (b) Evaluate the line integral using Green's theorem.

10. Find the surface area of

    (a) the surface obtained by rotating the curve $y = x^{1/2}, 0 \le x \le 1$ about the y-axis.

    (b) the tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

    (c) S is the surface described by $x = u - v, y = u + v, z = uv$ as u, v are in the disk of radius 2 centered at the origin in the uv-plane.

11. Exercise 9, page 472.

12. Find the tangent plane to the surface S at the point P in the following cases.

    (a) $S : x^2 + y^5 \cos(xy) - z^6 = -648, P = (9, 0, 3)$.

    (b) S is the graph of $f(x, y) = e^{2x^2y} + \sin(x + y), P = (-1, 1, f(-1, 1))$.

    (c) S is the cylinder of radius 5 centered around the y-axis and $P = (3, 19, 4)$.

13. Find the area of the following plane regions.

    (a) $x^{2/5} + y^{2/5} < 4$;

    (b) one loop of the four-leafed rose $r = 3 \sin \theta$.

14. Consider the rectangle R with vertices $(0, 0), (1, 0), (1, 4)$ and $(0, 4)$. The boundary of R is the curve C, consisting of C_1, the segment from $(0, 0)$ to $(1, 0)$; C_2, the segment from $(1, 0)$ to $(1, 4)$; C_3 the segment from $(1, 4)$ to $(0, 4)$; and C_4 the segment from $(0, 4)$ to $(0, 0)$. Consider the vector field $\vec{F} = (xy + \sin x \cos y)\hat{\mathbf{i}} - (\cos x \sin y)\hat{\mathbf{j}}$.

    (a) For each of C_i find a normal unit $\hat{\mathbf{n}}_i$ vector pointing out of the rectangle.

    (b) Find the flux of $\displaystyle\sum_{i=1}^{4} \int_{C_i} \vec{F} \cdot \hat{\mathbf{n}}_i ds$ out of R through C. Show your reasoning.

    (c) Is the total flux $\displaystyle\sum_{i=1}^{3} \int_{C_i} \vec{F} \cdot \hat{\mathbf{n}}_i ds$ out of R through C_1, C_2 and C_3, more than, less than or equal to the flux out of R through C? Show your reasoning.

15. The equations

$$z = 12, x^2 + y^2 \leq 25$$

describe a disk S of radius 5 lying in the plane $z = 12$. Compute the flux of $\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{i}} + z\hat{\mathbf{k}}$ across S.

16. Suppose a temperature function is given in space by $T(x, y, z) = x^2 + y^2 + z^2$ and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ with the outward normal. Find the heat flux across the surface S.

17. Compute the scalar curl and the divergence of the following plane vector fields.

    (a) $\vec{F} = 3xy\hat{\mathbf{i}} - \cos(x - y)\hat{\mathbf{j}}$

    (b) $\vec{F} = e^{x+2y}\hat{\mathbf{i}} + \arctan(xy)\hat{\mathbf{j}}$

18. Compute the vector curl $\nabla \times \vec{F}$ and the divergence of the following vector fields.

    (a) $\vec{F} = x\hat{\mathbf{i}} + e^{xy}\hat{\mathbf{k}}$

    (b) $\vec{F} = e^{x+2y}\hat{\mathbf{i}} + \arctan(xy)\hat{\mathbf{j}}$

    (c) $\vec{F} = \sqrt{1 + x^2}\hat{\mathbf{j}} - 2xyz\hat{\mathbf{i}} + \tan z\hat{\mathbf{k}}$