MATH 20E Lecture 1 - Monday, April 1, 2013

Disclaimer: The first few weeks will be mostly review of material from MATH 20C. We still have to go through it, because some of it was skipped in the various versions of 20C taught in the past few years. The aim of these first lectures is **not** to re-teach the concepts of multivariable calculus, but to refresh your memory and make sure everyone is on the same page with notation, etc...

Functions of several variables

Recall: for a function of 1 variable, we can plot its graph, and the derivative is the slope of the tangent line to the graph. Plotting graphs of functions of 2 variables: contour maps using level curves. Shown temperature map.

Contour map gives some qualitative info about how f varies when we change x, y.

The methods and concepts we develop will hold in n variables. But for simplicity, most of our examples will have only 2 or 3 variables.

Partial derivatives

Again this is recap from 20C.

$f_x = \dfrac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$; same for f_y.

Geometric interpretation: f_x, f_y are slopes of tangent lines of vertical slices of the graph of f (fixing $y = y_0$; fixing $x = x_0$).
How to compute: treat x as variable, y as constant.

In vector notation, for a function of n variables

$$\frac{\partial f}{\partial x_j}(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{e}_j)}{h}$$

where

$$e_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0).$$

Examples: Example: $f(x, y) = x^3 y + y^2)$. Then $\frac{\partial f}{\partial x} = 3x^2 y$ (treat y as a constant, x as a variable) and $\frac{\partial f}{\partial y} = x^3 + 2y$.

Example: $g(x, y) = \sin(x^3 y + y^2)$. Then $\frac{\partial g}{\partial x} = (3x^2 y)\cos(x^3 y + y^2)$ and $\frac{\partial g}{\partial y} = (x^3 + 2y)\cos(x^3 y + y^2)$.

Particular case $f : \mathbb{R}^n \to R$ the differential, which is a $1 \times n$ matrix, can be identified with the gradient vector

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \quad \dots \quad, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Example: $f(x, y, z) = x^2 + y^2 + z^3 \implies \nabla f = (2x, 2y, 3z^2)$. In particular $T = \nabla f(1, 2, 3) = (2, 4, 27)$ and it defines a linear map on R^3 given by the dot product. For instance,

$$T(-1, 1, 0) = (2, 4, 27) \cdot (-1, 1, 0) = -2.$$

Another example: $h(x,y) = \ln(x^2+y^2-xz)$. Then

$$\frac{\partial h}{\partial x} = \frac{2x-z}{x^2+y^2-xz}, \quad \frac{\partial h}{\partial y} = \frac{2y}{x^2+y^2-xz}, \quad \frac{\partial h}{\partial z} = \frac{-x}{x^2+y^2-xz}.$$

We can also take higher order partial derivatives. For instance,

$$\frac{\partial^2 h}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2-xz}\right) = \frac{(-1)(x^2+y^2-xz)-(-x)(2x-z)}{(x^2+y^2-xz)^2}.$$

Used one instance of the chain rule: if $g = F(u)$ and $u = u(x,y,z)$ then

$$\boxed{\frac{\partial g}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}}$$

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The *Jacobian matrix* of $f = (f_1, \dots, f_m)$ of n variables that takes values in $\mathbb{R}^m$ is also called the *differential (derivative)* of f. At a point $\vec{a} = (a_1, \dots, a_n)$ is given by

$$T = Df(\vec{a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \dots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix}$$

Note that it is an $m \times n$ matrix. $f(\vec{x})$ is differentiable at $\vec{a}$ if

$$\lim_{\vec{x}\to\vec{a}} \frac{\|f(\vec{x}) - f(\vec{a}) - T(\vec{x}-\vec{a})\|}{\|x-a\|} = 0,$$

where $T(\vec{x}-\vec{a})$ denotes the matrix multiplication between the $m\times n$ matrix T and the n-dimensional vector $(\vec{x}-\vec{a})$.

Example: $f(x,y,z) = (\sin(xyz), x^2+y^2-z)$ and $\vec{a} = (1,5,0)$. Then

$$Df = \begin{pmatrix} yz\cos(xyz) & xz\cos(xyz) & xy\cos(xyz) \\ 2x & 2y & -1 \end{pmatrix}$$

and

$$Df(\vec{a}) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 10 & -1 \end{pmatrix}$$

$$Df(\vec{a})(2,3,3) = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 10 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 31 \end{pmatrix}$$

Linear approximation

$$z = f(x,y)$$

Linear approximation formula: $\Delta f \approx f_x \Delta x + f_y \Delta y$.

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

where $z_0 = f(x_0,y_0)$.
Approximation formula = the graph is close to its tangent plane.

Example: $z = \cos x + e^{1-y^2}$ at $(x_0,y_0) = \left(\frac{\pi}{2},1\right)$. Then $z_0 = \cos\frac{\pi}{2} + e^0 = 1$ and $\frac{\partial z}{\partial x} = -\sin x$, $\frac{\partial z}{\partial y} = -2ye^{1-y^2}$. Thus $a = \frac{\partial z}{\partial x}(x_0,y_0) = \sin\frac{\pi}{2} = -1$ and $b = \frac{\partial z}{\partial y}(x_0,y_0) = -2$. The tangent plane has equation

$$z - 1 = -1\left(x - \frac{\pi}{2}\right) - 2(y-1) \iff x + 2y + z = 3 + \frac{\pi}{2}.$$

Properties of the differential:

- $f : \mathbb{R}^n \to \mathbb{R}^m$, c real number. The differential of $h = cf$ is $Dh(\vec{a}) = cDf(\vec{a})$.
- $f, g\mathbb{R}^n \to \mathbb{R}^m$. The differential of $h = f + g$ is $Dh(\vec{a}) = Df(\vec{a}) + Dg(\vec{a})$.
- (product rule) $f, g : \mathbb{R}^n \to \mathbb{R}$. The differential of $h = fg$ is $Dh(\vec{a}) = g(\vec{a})Df(\vec{a}) + f(\vec{a})Dg(\vec{a})$.
- (quotient rule) $f, g : \mathbb{R}^n \to \mathbb{R}$. The differential of $h = \frac{f}{g}$ is $Dh(\vec{a}) = \dfrac{g(\vec{a})Df(\vec{a}) - f(\vec{a})Dg(\vec{a})}{(g(\vec{a}))^2}$.

MATH 20E Lecture 3 - Friday, April 5, 2013

Chain rule

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$ and set $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$. Then, for $\vec{a} \in \mathbb{R}^n$ and $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$ we have

$$\boxed{D(g \circ f)(\vec{a}) = Dg(\vec{b})\, Df(\vec{a})} \text{ (matrix multiplication)}$$

Special cases:

- $\vec{c} : \mathbb{R} \to \mathbb{R}^3, c(t) = (x(t), y(t), z(t))$ path and $f : \mathbb{R}^3 \to R$. Then the derivative of $h(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$ is

  $$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = (\nabla f(\vec{c}(t))) \cdot (\vec{c}\,'(t)).$$

  Example: $\vec{c}(t) = (t, t^2, t^3)$, $f(x,y,z) = x^2 + y^2 - \cos z$. Chain rule gives

  $$\frac{dh}{dt} = (2x)1 + (2y)(2t) + (\sin z)(3t^2) = 2t + 4t^3 + et^2 \sin(t^3)$$

  where $h(t) = f(\vec{c}(t)) = t^2 + t^4 - \cos(t^3)$.

- $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}$ $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))\,, h(x,y,z) = g \circ f = g\,(u(x,y,z), v(x,y,z), w(x,y$

  $$\nabla h = \nabla g \, Df.$$

  i.e.

  $$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x}$$

  and so on.

  Example: $g(u,v,w) = u^2 + v^2 - w, u = x^2 y, v = y^2, w = e^{-xz}$.

  $$\frac{\partial h}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = (2u)(x^2) + (2v)(2y) + (-1)(-e^{-xz}) = 2x^4 y + 4y^3 + e^{-xz}$$

  where $h(x,y,z) = x^4 y^2 + y^4 - e^{-xz}$.

Taylor's formula

In 1 variable: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x,a)$ where the remainder has the property that

$$\lim_{x \to a} \frac{R_k(x,a)}{(x-a)^k} = 0.$$

Note that when $k = 1$ this boils down to the linear approximation formula.

Similarly, for a function $f : \mathbb{R}^n \to \mathbb{R}$ we have in the first two cases of the Taylor formula:

- **linear approximation** ($k = 1$)

  $$f(\vec{a} + \vec{h}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}) h_i + R_1(\vec{a}, \vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + R_1(\vec{a}, \vec{h})$$

  where

  $$\lim_{\vec{h} \to \vec{0}} \frac{R_1(\vec{a}, \vec{h})}{\|h\|} = 0.$$

- **quadratic approximation** ($k = 2$)

  $$f(\vec{a} + \vec{h}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a}) h_i + \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j + R_2(\vec{a}, \vec{h})$$

  where

  $$\lim_{\vec{h} \to \vec{0}} \frac{R_2(\vec{a}, \vec{h})}{\|h\|^2} = 0.$$

  In matrix form:

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + (h_1, \dots, h_n) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} f(\vec{a}) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} f(\vec{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} f(\vec{a}) & \dots & \frac{\partial^2 f}{\partial x_n^2} f(\vec{a}) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} + R_2(\vec{a}, \vec{h})$$

Example: $f(x,y) = \sin(x+2y)$ and $\vec{a} = (0,0) = \vec{0}$. Then

$$\nabla f = (\cos(x+2y), 2\cos(x+2y)$$

and the Hessian matrix of the second derivatives is

$$H = \begin{pmatrix} -\sin(x+2y) & -2\sin(x+2y) \\ -2\sin(x+2y) & -4\sin(x+2y) \end{pmatrix}$$

In particular, $f(\vec{a}) = 0$, $\nabla f(\vec{a}) = (1,2)$ and $H(\vec{a}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
The second order Taylor formula is

$$f(h_1, h_2) = h_1 + 2h_2 + R_2(\vec{0}, \vec{h}).$$