MATH 20E Lecture 27 - Monday, May 20, 2013

Why is Stokes' Theorem true?

Stokes' Theorem: If C is a *closed curve* in space, and S *any* surface bounded by C with *compatible orientation*, then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS}$$

"Proof" of Stokes: 1) if C and S are in the xy-plane then the statement follows from Green.

2) if C and S are in an arbitrary plane: this also reduces to Green in the given plane. Green/Stokes works in any plane because of geometric invariance of work, curl and flux under rotations of space. They can be defined in purely geometric terms so as not to depend on the coordinate system (x, y, z); equivalently, we can choose coordinates (u, v, w) adapted to the given plane, and work with those coordinates, the expressions of work, curl, flux will be the familiar ones replacing x, y, z with u, v, w.

3) in general, we can decompose S into small pieces, each piece is nearly flat (slanted plane); on each piece we have approximately work = flux by Greens theorem. When adding pieces, the line integrals over the inner boundaries cancel each other and we get the line integral over C; the flux integrals add up to flux through S.

Stokes and surface independence

In Stokes we can choose any surface S bounded by C : so if a same C bounds two surfaces S_1, S_2, then $\int_C \vec{F} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS$? Can we prove directly that the two flux integrals are equal?

Answer: change orientation of S_2, then $S = S_1 - S_2$ is a closed surface with $\hat{\mathbf{n}}$ pointing outwards; so we can apply the divergence theorem: $\iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = \iiint_W \operatorname{div}(\nabla \times \vec{F}) dV$. But $\operatorname{div}(\nabla \times \vec{F}) = 0$ always. (Checked by calculating in terms of components of $\vec{F}$.)

Applications of div and curl to physics

Recall: vector curl of velocity field = 2·angular velocity vector (of the rotation component of motion). E.g., for uniform rotation about z-axis, $\vec{v} = \omega(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$ and $\nabla \times \vec{v} = 2\omega\hat{\mathbf{k}}$.
Curl singles out the rotation component of motion (while div singles out the stretching component).

Interpretation of curl for force fields

If we have a solid in a force field (or rather an acceleration field!) $\vec{F}$ such that the force exerted on Δm at (x, y, z) is $F(x, y, z)\Delta m$: recall the torque of the force about the origin is defined as $\tau = \vec{r} \times \vec{F}$ and measures how $\vec{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

Hence: vector curl of $\dfrac{\text{Force}}{\text{mass}} = 2 \cdot \dfrac{\text{Torque}}{\text{moment of inertia}}$.

Consequence: if $\vec{F}$ derives from a potential, then $\nabla \times \vec{F} = \nabla \times (\nabla f) = 0$, so $\vec{F}$ does not induce any rotation motion. E.g., gravitational attraction by itself does not affect Earth's rotation. (not strictly true: actually Earth is deformable; similarly, friction and tidal effects due to Earth's gravitational attraction explain why the Moon's rotation and revolution around Earth are synchronous).

Div and curl of electrical field – part of Maxwells equations for electromagnetic fields.

Gauss-Coulomb law: $\operatorname{div} \vec{E} = \frac{\rho}{\epsilon_0}$ (ρ = charge density and ϵ_0 = physical constant).

By divergence theorem, can reformulate as: $\displaystyle\iint_S \vec{E} \cdot \hat{\mathbf{n}} dS = \iiint_W (\operatorname{div} \vec{E}) dV = \frac{Q}{\varepsilon_0}$, where Q = total charge inside the closed surface S.
This formula tells how charges influence the electric field; e.g., it governs the relation between voltage between the two plates of a capacitor and its electric charge.

MATH 20E Lecture 28 - Wednesday, May 22, 2013

Review for final exam - part I

vectors: dot product ($\vec{v} \cdot \vec{w} = \sum v_i w_i = \|\vec{v}\| \|\vec{w}\| \cos\theta$), cross product ($\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta = areaparalelogram$

functions of several variables
$f : \mathbb{R}^n \to \mathbb{R}$ $f(x, y, z, \dots)$
partial derivatives: $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z} \dots$
gradient vector: $\nabla f = (f_x, f_y, f_z, \dots)$
one example of chain rule:

- if $g = F(u)$ and $u = u(x, y, z)$ then $\frac{\partial g}{\partial x} = \frac{dF}{du} \frac{\partial u}{\partial x}$

More generally, the Jacobian matrix of $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ of n variables that takes values in $\mathbb{R}^m$ is given by

$$T = Df(\vec{a}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\vec{a}) & \dots & \dfrac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\vec{a}) & \dots & \dfrac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix}.$$

chain rule: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$ and set $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$. Then, for $\vec{a} \in \mathbb{R}^n$ and $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$ we have $D(g \circ f)(\vec{a}) = Dg(\vec{b})\, Df(\vec{a})$ (matrix multiplication)

Special cases:

- $\vec{c} : \mathbb{R} \to \mathbb{R}^3, c(t) = (x(t), y(t), z(t))$ path and $f : \mathbb{R}^3 \to R$. Then the derivative of $h(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$ is

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = (\nabla f(\vec{c}(t))) \cdot \left(\vec{c}'(t)\right).$$

- $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}$ $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))\,, h(x,y,z) = g \circ f = g\,(u(x,y,z), v(x,y,z), w($

linear approximation formula for $f(x, y, z, \ldots) : \Delta f \approx f_x \Delta x + f_y \Delta y + \ldots$

tangent planes to surfaces

- S is $z = f(x, y)$ the graph of $f(x, y)$
  Tangent plane at (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$ ihas equation $f_x(x - x_0) + f_y(y - y_0) = z - z_0$.
- S is the level surface $g(x, y, z) = 0$, then the tangent plane at (x_0, y_0, z_0) has equation $\nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$.
- S is parametrized by $\Phi(u, v)$
  normal vector: $\Phi_u \times \Phi_v$

double integrals: draw the region!

setup: need bounds of integration, then evaluate first inner integral and then outer.

$$\iint_R f(x,y)dA = \int_{x_{\min}}^{x_{\max}} \int_{y_{\text{bottom}}(x)}^{y_{\text{top}}(x)} f(x,y)dydx$$

polar coordinates: $x = r\cos\theta, y = r\sin\theta \implies dA = rdrd\theta$

general change of variables: $x = x(u, v), y = y(u, v) \implies dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$ (absolute value!)

MATH 20E Lecture 29 - Friday, May 24, 2013

Review for final exam - part II

triple integrals: setup: need bounds of integration then evaluate innermost integral and get a double integral; now do the double integral

$$\iiint_W f(x,y,z)dV = \iint_{\text{shadow in the } xy\text{-plane}} \left[\int_{z_{\text{bottom}}(x,y)}^{z_{\text{top}}(x,y)} f(x,y,z)dz\right] dA$$

rectangular coordinates: $dV = dxdydz$

cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z \implies dV = dzrdrd\theta$

spherical coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi \implies dV = \rho^2\sin\phi d\rho d\phi d\theta$

general change of variables: $x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \implies dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$

(absolute value again!)

vector fields: recall flow lines, how to sketch vector fields
work and line integrals: work $= \int_C \vec{F} \cdot d\vec{r}$ where C is a curve in plane, space, etc...

in 2D: $\vec{F} = (M, N) \implies \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$ (to evaluate: express everything in terms of a single parameter)

in 3D: $\vec{F} = (P, Q, R) \implies \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz$ (to evaluate: express everything in terms of a single parameter)

gradient fields and path independence:
If $\vec{F}$ is defined in a simply connected region (in plane or space) and $\nabla \times \vec{F} = 0$, then $\vec{F}$ is a gradient fields, i.e. $\vec{F} = \nabla g$ for some function $g(x, y)$ or $g(x, y, z)$.
To find potential: 2 methods
A. compute a line integral, e.g. $(0, 0)$ to $(x_1, 0)$ to (x_1, y_1)
B. antiderivatives
For gradient fields, work is given by the Fundamental Theorem of Calculus

$$\int_C \nabla g \cdot d\vec{r} = g(\text{end point}) - g(\text{start point}).$$

flux in plane: flux of $\vec{F} = (M, N)$ across a curve C in the plane is given by

$$\text{flux} = \int_C \vec{F} \cdot \hat{\mathbf{n}} ds$$

where $\hat{\mathbf{n}}$ is the unit normal pointing to the right of the curve (i.e. $\hat{\mathbf{T}}$ rotated 90° *clockwise*)
in coordinates $\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \int_C M dy - N dx = \int_C -N dx + M dy$ (to evaluate: same as before, since it is a line integral)
flux in space: flux of $\vec{F} = (P, Q, R)$ across a surface S in space is given by

$$\text{flux} = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S}$$

where $\hat{\mathbf{n}}$ is a unit normal (orientation might be specified or left to you to choose).

- S is $z = f(x, y)$ the graph of $f(x, y) \implies \hat{\mathbf{n}} dS = \pm(-f_x, -f_y, 1) dx dy$
- S is parametrized by $\Phi(u, v) \implies \hat{\mathbf{n}} dS = \pm \Phi_u \times \Phi_v du dv$
- if we know that $\vec{N}$ is a normal vector to the surface S, then $\hat{\mathbf{n}} dS = \pm \dfrac{\vec{N}}{\vec{N} \cdot \hat{\mathbf{k}}} dA$ (e.g. slanted plane; level surface $g(x, y, z) = 0$ and $\vec{N} = \nabla g$.)

	2D $\vec{F} = (M, N)$	3D $\vec{F} = (P, Q, R)$
work	**Green's Theorem:** $C =$ *closed* curve oriented counterclockwise enclosing region R $\int_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl}\, \vec{F}) dA$ in coordinates: $\int_C M dx + N dy = \iint_R (N_x - M_y) dA$	**Stokes' Theorem:** $C =$ curve in space $S =$ *any surface* bounded by C with compatible orientation $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS$ where $\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
flux	**Green's theorem (normal form):** C and R as above $\hat{\mathbf{n}}$ pointing outwards from R $\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \iint_R (\text{div}\, \vec{F}) dA$ in coordinates: $\int_C M dy - N dx = \iint_R (M_x + N_y) dA$	**Divergence theorem:** $S =$ closed surface enclosing solid W $\hat{\mathbf{n}}$ pointing outwards from R $\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iiint_W (\text{div}\, \vec{F}) dV$ where $\text{div}\, \vec{F} = P_x + Q_y + R_z$

Have a nice summer!