MATH 20E Lecture 4 - Monday, April 8, 2013

More recap from MATH 20C.

Double integrals

$$\iint_R f(x,y)dA, \quad dA = dxdy = dydx$$

We compute by reducing to an iterated integral

$$\iint_R f(x,y)dA = \int_{x_{\min}}^{x_{\max}} S(x)dx, \quad \text{where } S(x) = \int_{y_{\min}(x)}^{y_{\max}(x)} f(x,y)dy \text{ for each } x$$

Example 1 $f(x,y) = 1 - x^2 - y^2$ and $R : 0 \le x \le 1, 0 \le y \le 1$.

$$\int_0^1 \int_0^1 \left(1 - x^2 - y^2\right) dy\, dx$$

How to evaluate?
1) inner integral (x is constant):

$$\int_0^1 \left(1 - x^2 - y^2\right) dy = \left[y - x^2 y - \frac{y^3}{3}\right]_{y=0}^{y=1} = \left(1 - x^2 - \frac{1}{3}\right) - 0 = \frac{2}{3} - x^2.$$

2)outer integral: $\displaystyle\int_0^1 \left(\frac{2}{3} - x^2\right) dx = \left[\frac{2}{3}x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Example 2 Same function over the quarter-disk $R : x^2 + y^2 \le 1, 0 \le x \le 1, 0 \le y \le 1$.
How to find the bounds of integration? Fix x constant and look at the slice of R parallel to y-axis. Bounds from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is at $x = 0$, last slice is at $x = 1$. So we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy\, dx.$$

Note that the inner bounds depend on the outer variable x; the outer bounds are constants!
1) inner integral (x is constant):

$$\int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy = \left[(1 - x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}} = (1 - x^2)^{3/2} - \frac{(1 - x^2)^{3/2}}{3} = \frac{1}{3}(1 - x^2)^{3/2}.$$

2)outer integral:

$$\int_0^1 \frac{1}{3}(1 - x^2)^{3/2} dx = \ldots (\text{trig substitution } x = \sin\theta, \text{ double angle formulas}) \ldots = \frac{\pi}{8}.$$

This is complicated! It will be easier to do it in polar coordinates.

Example 3 $\int_0^1 \int_y^{\sqrt{y}} \frac{e^x}{x} dx\, dy$ (Inner integral has no formula.)
To exchange order: 1) draw the region (here: $y \le x \le \sqrt{y}$ for $0 \le y \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of x, what are the bounds for y? Picture: left border is $y = x$, right is $x^2 = y$; first slice is $x = 0$, last slice is $x = 1$, so we get

$$\int_0^1 \int_{x^2}^x \frac{e^x}{x} dy\, dx = \int_0^1 \frac{e^x}{x}(x - x^2) dx = \int_0^1 e^x(1-x) dx \overset{\text{parts}}{=} [e^x(1-x)]_{x=0}^{x=1} + \int_0^1 e^x dx = e - 2.$$

Example 4 Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \le x \le 2$.

Both surfaces look like parabola-shaped tunnels along the x-axis. They intersect at $1 - y^2 = y^2 - 1 \implies y = \pm 1$. So $z = 0$ and x can be anything, therefore lines parallel to the x-axis (picture drawn). Get volume by integrating the difference $z_{\text{top}} - z_{\text{bottom}}$, i.e. take the volume under the top surface and subtract the volume under the bottom surface (same idea as in 1 variable).

$$\begin{aligned} \text{vol} &= \int_0^2 \int_{-1}^1 \big((1-y^2) - (y^2-1)\big)\, dy\, dx = 2\int_0^2 \int_{-1}^1 (1-y^2) dy\, dx \\ &= 2\int_0^2 \left[y - \frac{y^3}{3}\right]_{y=-1}^{y=1} dx = 2\int_0^2 \frac{4}{3} dx = \frac{16}{3}. \end{aligned}$$

Triple integrals

$$\iiint_R f(x,y,z)\, dV \quad (R \text{ is a solid in space})$$

Note: $\Delta V = \text{area(base)} \cdot \text{height} = \Delta A \Delta z$, so $dV = dA\, dz = dx\, dy\, dz$ or any permutation of the three.

Example 1 $R:$ the region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$. (picture drawn)

The volume of this region is $\iiint_R 1\, dV = \iint_D \left[\int_{x^2+y^2}^{4-x^2-y^2} dz\right] dA$, where D is the shadow in the xy-plane of the region R.

To set up bounds, (1) for fixed (x, y) find bounds for z : here lower limit is $z = x^2 + y^2$, upper limit is $z = 4 - x^2 - y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $\ge z$ on bottom surface? Answer: when $4 - x^2 - y^2 \ge x^2 + y^2$, i.e. $x^2 + y^2 \le 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get

$$\text{vol}(R) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\, dy\, dx.$$

Actual evaluation would be easier using polar coordinates.

MATH 20E Lecture 5 - Wednesday, April 10, 2013

Discussed Examples 5 and 6 from Section 5.5.

MATH 20E Lecture 6 - Friday, April 12, 2013

Example 1 area of ellipse with semiaxes a and b : setting $u = x/a, v = y/b$,

$$\iint_{(x/a)^2+(y/b)^2<1} dxdy = \iint_{u^2+v^2<1} ab\ dudv = ab \iint_{u^2+v^2<1} dudv = \pi ab.$$

(substitution works here as in 1-variable calculus: $du = \frac{1}{a}dx, dv = \frac{1}{b}dy$, so $dudv = \frac{1}{ab}dxdy$.)

In general, must find out the scale factor (ratio between $dudv$ and $dxdy$).

Example 2 set $u = 3x - 2y, v = x + y$ to simplify either integrand or bounds of integration. What is the relation between $dA = dxdy$ and $dA^* = dudv$? (area elements in xy- and uv-planes).

Answer: consider a small rectangle of area $\Delta A = \Delta x \Delta y$, it becomes in uv-coordinates a parallelogram of area ΔA^*. Here the answer is independent of which rectangle we take, so we can take for instance the unit square in xy-coordinates.

We have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So the unit square in the xy-plane becomes a parallelogram in the uv-plane with sides given by the vectors $(3, 1)$ and $(-2, 1)$. (Picture drawn.) The are of the parallelogram is given by the the absolute value of the determinant

$$\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5 \left(= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \right).$$

For any rectangle $\Delta A^* = 5\Delta A$ and in the limit $dA^* = 5dA$, i.e. $dudv = 5dxdy$. So

$$\iint \ldots dxdy = \iint \ldots \frac{1}{5} dudv.$$

General case: If $u = u(x, y), v = v(x, y)$ is our change of variable, the approximation formula says that $\Delta u \approx u_x \Delta x + u_y \Delta y, \Delta v \approx v_x \Delta x + v_y \Delta y$. Hence

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}.$$

A small xy-rectangle is approx. a parallelogram in uv-coords, but scale factor depends on x and y now. By the same argument as before, the scale factor is the determinant.

Definition: the Jacobian is $J = \frac{\partial(u,v)}{\partial(x,y)} \stackrel{\text{def}}{=} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$.

Then

$$\boxed{dudv = |J| dxdy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy}$$

(absolute value because area is the absolute value of the determinant)

Example 3: polar coordinates $x = r\cos\theta, y = r\sin\theta$:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Since $r \geq 0$, get $dxdy = rdrd\theta$.