MATH 20E Lecture 7 - Monday, April 15, 2013

Example 1: compute the integral $\iint_{x^2+y^2\le 1, x\ge 0, y\ge 0}(1-x^2-y^2)dxdy$ from Lecture 4 using polar coordinates $x = r\cos\theta, y = r\sin\theta$ (picture drawn). We have seen last time that $dxdy = rdrd\theta$. In polar coordinates the quarter-disk becomes $0 \le \theta \le \pi/2, 0 \le r \le 1$. Putting it all together

$$\iint_{x^2+y^2\le 1, x\ge 0, y\ge 0}(1-x^2-y^2)dxdy = \int_0^{\pi/2}\int_0^1 (1-r^2)rdrd\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{\pi}{8}.$$

Example 2: compute $\int_0^1\int_0^1 x^2y\,dxdy$ by changing to $u = x, v = xy$ (usually motivation is to simplify either integrand or region; here neither happens, but we just illustrate the general method).

1. Area element: Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x.$$

   Therefore $dudv = |x|dxdy = xdxdy$ (since in our region $x \ge 0$). Get $dxdy = \frac{1}{x}\,dudv$

2. Express integrand in terms of $u, v: x^2y\,dxdy = x^2y\frac{1}{x}dudv = xy\,dudv = v\,dudv$.

3. Find bounds (picture drawn): if we integrate $dudv$, then first we keep $v = xy$ constant, slice looks like portion of hyperbola (picture shown), parametrized by $u = x$. The bounds are: at the top boundary $y = 1$, so $v/u = 1$, i.e. $u = v$; at the right boundary, $x = 1$, so $u = 1$. So the inner integral is $\int_v^1$. The first slice is $v = 0$, the last is $v = 1$; so we get

$$\int_0^1\int_v^1 v\,dudv.$$

Besides the picture in xy-coordinates (a square sliced by hyperbolas), I also drew a picture in uv-coordinates (a triangle), which some students may find is an easier way of getting the bounds for u and v.

Evaluate the integral

$$\int_0^1\int_v^1 v\,dudv = \int_0^1 v(1-v)\,dv = \frac{1}{6}.$$

MATH 20E Lecture 8 - Wednesday, April 17, 2013

Change of variables for triple integrals

If $u = u(x,y,z), v = v(x,y,z), w = w(x,y,z)$ is our change of variables, we use the same argument as in two dimensions shows to figure out the ration between $dudvdw$ and $dxdydz$. The Jacobian of the transformation is

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} \stackrel{\text{def}}{=} \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

Then

$$\boxed{dudvdw = |J|dxdydz = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| dxdy}$$

(absolute value because area is the absolute value of the determinant)

Cylindrical coordinates

(r, θ, z) with $x = r\cos\theta, y = r\sin\theta$ and $z = z$ (unchanged); r measures distance from z-axis, θ measures angle from xz-plane (picture drawn).
Cylinder of radius a centered on z-axis is $r = a$ (drawn); $\theta = \pi/3$ is a vertical half-plane (drawn).
Compute the Jacobian and get $dxdydz = rdrd\theta dz$.

Example $R =$ region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ (picture drawn). Want to compute the volume of the solid, i.e. $\iiint_R dV$.

To set up bounds:

1. for fixed (x, y) find bounds for z : here lower limit is $z = x^2 + y^2$, upper limit is $z = 4 - x^2 - y^2$;

2. find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $>$ z on bottom surface? Answer: when $4 - x^2 - y^2 > x2 + y2$, i.e. $x^2 + y^2 < 2$. (disk of radius $\sqrt{2}$)

In cylindrical coordinates, we get $\iiint_R dxdydz = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r\, dzdrd\theta = \ldots = 4\pi$.

Spherical coordinates

(ρ, ϕ, θ)
$\rho =$ rho $=$ distance to origin ≥ 0
$\phi =$ phi $=$ angle down from positive z-axis, $0 \leq \phi \leq \pi$
$\theta =$ same as in cylindrical coordinates, $0 \leq \theta \leq 2\pi$

Diagram drawn in space, and picture of 2D slice by vertical plane with z, r coordinates.

Formulas to remember: $z = \rho\cos\phi, r = \rho\sin\phi$ so $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$.
$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$. On the surface of the sphere, ϕ is similar to latitude, except it's 0 at the north pole, $\pi/2$ on the equator, π at the south pole; θ is similar to longitude.

Jacobian $J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi \geq 0$, so $dxdydz = \rho^2 \sin\phi\, d\rho d\phi d\theta$.

MATH 20E Lecture 9 - Friday, April 19, 2013

Vector fields: $\vec{F}$ assigns to a point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ a *vector* $\vec{F}(x_1, \ldots, x_n)$.

Examples: velocity fields, e.g. wind flow (shown: chart of Santa Ana winds and hurricane winds); force fields, e.g. gravitational field.

We will mostly be concerned with vector fields

- in 2D, i.e. for $n = 2$: $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}} = (M(x, y), N(x, y))$ (wind, velocity of motion in the plane); at each point in the plane we have a vector $\vec{F}$ which depends on x, y.

- in 3D, i.e. for $n = 3 : \vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ (gravitational field, velocity of motion is space); at each point in space we have a vector $\vec{F}$ which depends on x, y, z.

Examples drawn on blackboard (all in the plane): (1) $\vec{F} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (constant vector field); (2) $\vec{F} = x\hat{\mathbf{i}}$; (3) $\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ (radially outwards); (4) $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ (explained using that the vector $(-y, x)$ is the vector (x, y) rotated $90°$ counterclockwise) - velocity field for uniform rotation.

Gradient vector fields: $\vec{F} = \nabla f = f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}}$ for some function $f(x,y)$ (called the potential of the vector field)

Observe: if $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$ is a gradient field then $N_x = M_y$. Indeed, if $\vec{F} = \nabla f$ then $M = f_x$ and $N = f_y$, so $N_x = f_{yx} = f_{xy} = M_y$.

Claim: Conversely, if $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$ is defined and differentiable at every point of the plane, and $N_x = M_y$, then $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$ is a gradient field.

Example: $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} : N_x = 1, M_y = -1$ so $\vec{F}$ is not a gradient field.

Example: $\vec{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} : N_x = 1 = M_y$ and $\vec{F}$ is defined everywhere. So $\vec{F}$ is a gradient field. How to find potential $f(x,y)$? We need $f_x = y$ and $f_y = x$. Integrate $f_x = y$ with respect to the variable x (treat y as a constant) and get $f(x,y) = xy + c(y)$. Take derivative with respect to y and get $f_y = x + c'(y)$. But we have $f_y = x$, so $c'(y) = 0$, i.e. $c(y) =$ constant. Thus $f(x,y) = xy +$ const.

Flow lines: A flow line for a 3D vector field $\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ is a path $\vec{c}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ such that $\vec{F}(c(t)) = \vec{c}'(t)$ i.e. $\vec{F}(c(t))$ is tangent to the curve $\vec{c}$ at time t. (Concept similar in n-dimensional space). Shown computer demo.

Example: $\vec{F} = \hat{\mathbf{i}} + 2x\hat{\mathbf{j}} + 3y\hat{\mathbf{k}} = (1, 2x, 3y)$ and $c(t) = (t, t^2, t^3)$. Then $c'(t) = (1, 2t, 3t^2) = \vec{F}(c(t))$. Hence $c(t)$ is a flow line for $\vec{F}$.