MATH 20E Lecture 10 - Monday, April 22, 2013

Review for Midterm 1

Topics: functions of several variables, partial derivatives, chain rule, approximation formula, tangent planes; derivative matrix;
integration in several variables, change of variables
vector fields

Discussed problems 5 (partial derivatives, gradient, tangent plane and approximation formula), 6 (chain rule), 11 (triple integrals, spherical coordinates), 12 (general changes of variables) from the study guide.

Recall for general changes of variables: $u = u(x, y), v = v(x, y)$. The Jacobian is $J = \frac{\partial(u,v)}{\partial(x,y)} \stackrel{\text{def}}{=} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Then $dudv = |J|dxdy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy$ (absolute value because area is the absolute value of the determinant).

MATH 20E Lecture 11 - Wednesday, April 24, 2013: first midterm

MATH 20E Lecture 12 - Friday, April 26, 2013

Path integrals

Let $\vec{c}(t)$ be a path in n-dimensional space and f a function of n variables. We can integrate f along $\vec{c}$ and compute

$$\int_{\vec{c}} f ds.$$

Here ds stands for the arc length element as we have to cut the curve c into small pieces and measure their length. In 2 variables, this means computing the area of a fence that follows the path $\vec{c}(t)$ and at every point has height equal to $f(\vec{c}(t))$. (Picture drawn).

To evaluate: $ds = \|\vec{c}'(t)\|dt$ since $\|\vec{c}'(t)\|$ is the speed of a particle moving on c and distance = speed · time. So

$$\int_{\vec{c}} f ds = \int_a^b f(\vec{c}(t))\|\vec{c}'(t)\|dt.$$

Example (3 variables): $f(x, y, z) = x + y + z$ and $\vec{c}(t) = (\cos t, \sin t, t)$ with $0 \leq t \leq 2\pi$ (helix). Then $\vec{c}'(t) = (-\sin t, \cos t, 1)$ and $\|\vec{c}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$.

$$\int_{\vec{c}} f ds = \int_0^{2\pi} f(\cos t, \sin t, t)\sqrt{2}dt = \sqrt{2}\int_0^{2\pi} (\cos t + \sin t + t)dt = \sqrt{2}\left[\sin t - \cos t + \frac{t^2}{2}\right]_{t=0}^{t=2\pi} = 2\sqrt{2}\,\pi.$$

Work and line integrals

$W = (\text{force}) \cdot (\text{distance}) = \vec{F} \cdot \Delta\vec{r}$ for a small motion $\Delta\vec{r}$. Total work is obtained by summing these along a trajectory C : get a "line integral"

$$W = \int_C \vec{F} \cdot d\vec{r} \left(= \lim_{\Delta\vec{r} \to 0} \sum_i \vec{F} \cdot \Delta\vec{r}_i \right).$$

To evaluate the line integral, we observe C is parametrized by time t, with $a \leq t \leq b$ and give meaning to the notation $\int_C \vec{F} \cdot d\vec{r}$ by

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt.$$

Example: $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ and C is given by $x = t, y = t^2, 0 \leq t \leq 1$ (portion of parabola $y = x^2$ from $(0,0)$ to $(1,1)$). Then we substitute expressions in terms of t everywhere:

$$\vec{F} = (-y, x) = (-t^2, t), \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (1, 2t),$$

so $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_0^1 (-t^2, t) \cdot (1, 2t) dt = \int_0^1 t^2 dt = \frac{1}{3}$. (In the end things always reduce to a one-variable integral.)

New notation for line integral: $\vec{F} = (M, N)$, and $d\vec{r} = (dx, dy)$ (this is in fact a differential: if we divide both sides by dt we get the component formula for the velocity $d\vec{r}/dt$). So the line integral becomes

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy.$$

The notation is dangerous: this is not a sum of integrals w.r.t. x and y, but really a line integral along C. To evaluate one must express everything in terms of the chosen parameter.

In the above example, we have $x = t, y = t^2$, so $dx = dt, dy = 2t dt$; then

$$\int_C -y dx + x dy = \int_0^1 -t^2 dt + t(2t dt) = \int_0^1 t^2 dt = \frac{1}{3}.$$

(same calculation as before, using different notation).

In fact, the definition of the line integral does not involve the parametrization: so the result is the same no matter which parametrization we choose. For example we could choose to parametrize the parabola by $x = \sin\theta, y = \sin^2\theta, 0 \leq \theta \leq \pi/2$. Then we'd get $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \dots d\theta$ would be equivalent to the previous one under the substitution $t = \sin\theta$ and would again be equal to $1/3$. In practice we always choose the simplest parametrization!