MATH 20E Lecture 16 - Monday, May 6, 2013

Surface area continued

Last week, we have seen how to compute the surface area in 2 cases:

1) Parametric surface $S : x = x(u, v), y = y(u, v), z = z(u, v)$ with $(u, v) \in R$ some region of the uv-plane.

2) The surface S is the graph of a function $f(x, y)$ with $(x, y) \in R$ some region of the uv-plane.

Today, we we'll talk about two more cases:

3) Surface of revolution: S is obtained by taking the graph $y = f(x), a \leq x \leq b$ of a function of one variable and rotate around the x-axis (picture drawn).

$$\text{area}(S) = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\, dx$$

A small slice is a cylinder of height ds (arc length element) and with base a circle of radius $|f(x)|$. The length of the circle is $2\pi|f(x)|$ and $ds = \sqrt{1 + (f'(x))^2}\, dx$ (from MATH 20C).

Example: $f(x) = x, 0 \leq x \leq 1$. (picture drawn) Get cylinder of height 1, base circle of radius 1. Then the area is $2\pi \int_0^1 x\sqrt{2}\, dx = \pi\sqrt{2}$. (Same as last time).

4) Implicit surface $S : g(x, y, z) = 0$

For a slanted plane $ax + by_+cz = d$, the normal vector is $\mathbf{N} = (a, b, c)$. Picture drawn. Surface element $\Delta S =$? Look at projection to xy-plane: $\Delta A = \Delta S|\cos\alpha| = (|\mathbf{N} \cdot \hat{\mathbf{k}}|/\|\mathbf{N}\|)\Delta S$ (where $\alpha =$ angle between slanted surface element and horizontal: projection shrinks one direction by factor $|\cos\alpha| = (|\mathbf{N} \cdot \hat{\mathbf{k}}|)/\|\mathbf{N}\|$, preserves the other).

Hence $dS = \dfrac{\|\mathbf{N}\|}{|\mathbf{N} \cdot \hat{\mathbf{k}}|}dxdy$. For a general implicit surface S given by equation $g(x, y, z) = 0$ we use linear approximation. Normal vector to the surface is $\mathbf{N} = \nabla g$. Thus $dS = \dfrac{\|\nabla g\|}{|g_z|}dxdy$ and

$$\text{area}(S) = \iint_S dS = \iint_R \frac{\|\nabla g\|}{|g_z|}\, dxdy = \iint_R \sqrt{\frac{g_x^2 + g_y^2 + g_z^2}{g_z^2}}\, dxdy$$

where R is the shadow of S on the xy-plane.

Note: if S is vertical then the denominator is zero, cant project to xy-plane any more (but one could project e.g. to the xz-plane).

Example: $S =$ sphere of radius a. Given by implicit equation $x^2 + y^2 + z^2 - a^2 = 0$. Then $g(x, y, z) = x^2 + y^2 + z^2 - a^2$ and $\nabla g = (2x, 2y, 2z)$, which has length $2a$. On the other hand, $g_z = 2z = \pm 2\sqrt{a^2 - x^2 - y^2}$. The shadow on xy-plane is the disk of radius a, and we have

$$\text{area}(S) = 2\iint_{x^2+y^2\leq a^2} \frac{2a}{2\sqrt{a^2 - x^2 - y^2}}dxdy = 2a\int_0^a \int_0^{2\pi} \frac{1}{\sqrt{a^2 - r^2}}rd\theta dr = 4\pi a\int_0^a \frac{r}{\sqrt{a^2 - r^2}}dr.$$

Use substitution $u = a^2 - r^2$ and get that area of the sphere is equal to $4\pi a^2$.

Integrating scalar function on surfaces

$$\iint_S f(x,y,z)dS$$

1) Parametric surface $S: x = x(u,v), y = y(u,v), z = z(u,v)$ with $(u,v) \in R$ some region of the uv-plane.

The parametrization is $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ and we know that the area element is $dS = \|\Phi_u \times \Phi_v\| dudv$. Then

$$\iint_S f(x,y,z)dS = \iint_R f(x(u,v), y(u,v), z(u,v))\|\Phi_u \times \Phi_v\| dudv.$$

Example: Integrate $f(x,y,z) = \sqrt{x^2+y^2+1}$ on the surface $S: x = r\cos\theta, y = r\sin\theta, z = \theta, 0 \le r \le 1, 0 \le \theta \le 2\pi$ (helicoid, see picture on page 464 of the textbook).

$$\Phi_r \times \Phi_\theta = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -r\cos\theta & r\sin\theta & 1 \end{vmatrix} = \begin{vmatrix} \sin\theta & 0 \\ r\sin\theta & 1 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} \cos\theta & 0 \\ -r\cos\theta & 1 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} \cos\theta & \sin\theta \\ -r\cos\theta & r\sin\theta \end{vmatrix} \hat{\mathbf{k}} = \sin\theta\hat{\mathbf{i}} - \cos\theta\hat{\mathbf{j}} + r\hat{\mathbf{k}}.$$

Hence $\|\Phi_r \times \Phi_\theta\| = \sqrt{1+r^2}$ and $dS = \sqrt{1+r^2}drd\theta$. Therefore

$$\iint_S f(x,y,z)dS = \iint_S \sqrt{x^2+y^2+1}dS = \iint_S \sqrt{r^2+1}dS = \int_0^{2\pi}\int_0^1 (r^2+1)drd\theta = 8\pi/3.$$

2) The surface S is the graph of a function $g(x,y)$ with $(x,y) \in R$ some region of the uv-plane. Then $dS = \sqrt{1+g_x^2+g_y^2}\,dxdy$ and

$$\iint_S f(x,y,z)dS = \iint_R f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dxdy.$$

MATH 20E Lecture 17 - Wednesday, May 8, 2013

Flux in 3D

$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ where P, Q, R are functions of x, y, z S = surface in space

If $\vec{F}$ = velocity of a fluid flow, then flux = flow per unit time across surface S.

Cut S into small pieces, then over each small piece: what passes through ΔS in unit time is the contents of a parallelepiped with base ΔS and third side given by $\vec{F}$.

Volume of box = height $\times$ area of base = $(\vec{F} \cdot \hat{\mathbf{n}})\Delta S$ where $\hat{\mathbf{n}}$ is a unit normal vector to S.

Remark: there are 2 choices for $\hat{\mathbf{n}}$ (choose which way is counted positively = "orientation")

Notation: $d\vec{S} = \hat{\mathbf{n}}dS$ ($d\vec{S}$ is often easier to compute than $\hat{\mathbf{n}}$ and dS separately!).

In 3D, flux of a vector field is the double integral

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S}.$$

Example 1: $\vec{F} = (x, y, z)$ through sphere of radius a centered at 0.

$\hat{\mathbf{n}} = \frac{1}{a}(x, y, z)$ (other choice: $-\frac{1}{a}(x, y, z)$; traditionally choose $\hat{\mathbf{n}}$ pointing out).
$\vec{F} \cdot \hat{\mathbf{n}} = (x, y, z) \cdot \hat{\mathbf{n}} = \frac{1}{a}(x^2 + y^2 + z^2) = a$, so

$$\iint_S F \cdot \hat{\mathbf{n}} dS = \iint_S a dS = a(4\pi a^2).$$

Example 2: Same sphere, $\vec{H} = z\hat{\mathbf{k}}$ Then $\vec{H} \cdot \hat{\mathbf{n}} = \frac{z^2}{a}$ and

$$\iint_S \vec{H} \cdot n dS = \iint_S \frac{z^2}{a} dS$$

Parametrize S by $x = a\cos\theta\sin\phi, y = a\sin\theta\sin\phi, z = a\cos\phi$ with $0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Then

$$dS = \sqrt{\left(\frac{\partial(y,z)}{\partial(\theta,\phi)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(\theta,\phi)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(\theta,\phi)}\right)^2} \, d\theta d\phi = a^2 \sin\phi \; d\theta d\phi.$$

Flux is given by

$$\iint_S \vec{H} \cdot n dS = \iint_S \frac{z^2}{a} dS = \int_0^\pi \int_0^{2\pi} \frac{(a\cos\phi)^2}{a} a^2 \sin\phi \; d\theta d\phi = 2\pi a^3 \int_0^\pi \cos^2\phi \sin\phi d\phi = \frac{4\pi a^3}{3}.$$

Setup. Sometimes we have an easy geometric argument, but in general we must compute the surface integral. The setup requires the use of two parameters to describe the surface, and $\vec{F} \cdot \hat{\mathbf{n}} dS$ must be expressed in terms of them. How to do this depends on the type of surface.

1. S = parametric surface with parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$ $(u, v) \in R$ some region of the uv-plane.
   normal vector to the surface: $\Phi_u \times \Phi_v$, so unit normal $\hat{\mathbf{n}} = \dfrac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}$
   surface area element $dS = \|\Phi_u \times \Phi_v\| dudv$

   Hence $d\vec{S} = \hat{\mathbf{n}} dS = (\Phi_u \times \Phi_v) dudv$ and

   $$\boxed{\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (\Phi_u \times \Phi_v) dudv.}$$

2. S = graph of a function $g(x, y)$ with x, y in some region R of the xy-plane.

normal vector to the surface: $(-g_x, -g_y, 1)$ so unit normal $\hat{\mathbf{n}} = \dfrac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$

surface area element $dS = \sqrt{1 + g_x^2 + g_y^2}\; dA$

Hence $d\vec{S} = \hat{\mathbf{n}} dS = (-g_x, -g_y, 1) dA$ and

$$\boxed{\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (-g_x, -g_y, 1) dA.}$$

3. $S =$ implicit surface given by equation $f(x, y, z) = 0$.

   normal vector to the surface: ∇f, so unit normal $\hat{\mathbf{n}} = \dfrac{\nabla f}{\|\nabla f\|}$.

   area element $dS = \dfrac{\|\nabla f\|}{|\nabla f \cdot \hat{\mathbf{k}}|} dA$.

   Hence $d\vec{S} = \hat{\mathbf{n}} dS = \frac{1}{\nabla f \cdot \hat{\mathbf{k}}} \nabla f \, dA$ and

$$\boxed{\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (\nabla f) \frac{1}{\nabla f \cdot \hat{\mathbf{k}}} dA,}$$

   where R is the shadow of S on the xy-plane.

MATH 20E Lecture 18 - Friday, May 10, 2013

Scalar curl

$\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$ where M, N are functions of x, y.

curl$(\vec{F}) = N_x - M_y$ measures the failure of $\vec{F}$ to be conservative.

We have seen : $N_x = M_y \overset{*}{\iff} \vec{F}$ is a gradient field $\iff \vec{F}$ is conservative (i.e. $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C.)

$(*) : \implies$ only holds if $\vec{F}$ is defined everywhere, or in a simply-connected region (no holes).

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.

Example: $\vec{F} = (a, b)$ uniform translation; curl $\vec{F} = 0$

Example: $\vec{F} = (x, y)$ expanding motion has curl zero.

Example: $\vec{F} = (-y, x)$ rotation at unit angular velocity has curl $\vec{F} = 2$.

For a force field, curl $\vec{F}$ = torque exerted on a test mass, measures how $\vec{F}$ imparts rotation motion.

Green's Theorem

If C is a positively oriented (i.e. counterclockwise) closed curve enclosing a region R, then the work done by a vector field $\vec{F} = (M, N)$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\operatorname{curl} \vec{F}) dA \text{ which means } \int_C M dx + N dy = \iint_R (N_x - M_y) dA.$$

Application: proof of our criterion for gradient fields.

Theorem: if $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$ is defined and continuously differentiable in the whole plane, then $N_x = M_y \implies \vec{F}$ is conservative ($\vec{F}$ is a gradient field).

Proof: If $N_x = M_y$ then, by Green,$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\operatorname{curl} \vec{F}) dA = \iint_R 0 dA = 0$. So $\vec{F}$ is conservative.

Note: this only works if $\vec{F}$ and its curl are defined everywhere inside R.

Example: $\vec{F} = (x, y)$ and C = circle of radius a oriented counterclockwise. We have seen $\operatorname{curl} \vec{F} = 0$. To check that Green's theorem works, compute $\int_C \vec{F} \cdot d\vec{r}$.
Parametrization: $x = a\cos\theta, y = a\sin\theta$ $0 \le \theta \le 2\pi$. So $dx = -a\sin\theta d\theta, dy = a\cos\theta d\theta$

$$\int_C x dx + y dy = \int_0^{2\pi} -a^2 \sin\theta\cos\theta + a^2 \sin\theta\cos\theta d\theta = 0.$$

Example: $\vec{F} = (-y, x)$; have seen $\operatorname{curl} \vec{F} = 2$. Then for any closed curve C that encloses region R and is oriented counterclockwise. Plugging into $\int_C \vec{F} d\vec{r} = \iint_R (\operatorname{curl} \vec{F}) dA = 2 \iint_R dA$ get

$$\boxed{\operatorname{area}(R) = \frac{1}{2} \int_C x dy - y dx}$$

Example (reduce a complicated area integral to an easy line integral): example 2, page 525.

Example (reduce a complicated line integral to an easy $\iint$): Let C = unit circle centered at $(2, 0)$, counterclockwise. R = unit disk at $(2, 0)$. Then

$$\int_C ye^{-x} dx + \left(\frac{1}{2}x^2 - e^{-x}\right) dy = \iint_R N_x - M_y dA = (x + e^{-x}) - e^{-x} dA = \iint_R x dA.$$

Parametrize R in polar coordinates $x = 2 + r\cos\theta, y = r\sin\theta, 0 \le \theta \le 2\pi, 0 \le r \le 1$ and get

$$\iint_R x dA = \int_0^{2\pi} \int_0^1 (2 + r\cos\theta) r dr d\theta = \int_0^{2\pi} 1 + \frac{\cos\theta}{3} d\theta = 2\pi.$$

(Note: direct calculation of the line integral would probably involve setting $x = 2 + \cos\theta, y = \sin\theta$, but then we have exponential of trig functions and calculations get really complicated.)

Vector curl - only in 3D

Del operator. $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ (symbolic notation!)

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ gradient

$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = (P, Q, R)$ where P, Q, R are functions of x, y, z

Definition: vector curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial(Q,R)}{\partial(y,z)}\right)\hat{\mathbf{i}} - \left(\frac{\partial(P,R)}{\partial(x,z)}\right)\hat{\mathbf{j}} + \left(\frac{\partial(P,Q)}{\partial(x,y)}\right)\hat{\mathbf{k}}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

Note: If $\vec{F} = M(x,y)\hat{\mathbf{i}} + N(x,y)\hat{\mathbf{j}}$ is a plane vector field, we can think of it in space as $\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$. In this case, $\nabla \times \vec{F} = (\operatorname{curl} \vec{F})\hat{\mathbf{k}}$.