MATH 20E Lecture 19 - Monday, May 13, 2013

Taught by Prof. J. Roberts. Covered Green's Theorem and Stokes' theorem.

MATH 20E Lecture 20 - Wednesday, May 15, 2013

Review for the second Midterm

Work in space and plane

Work of a vector field $\vec{F}$ (could be 2-dimensional, 3-dimensional or n-dimensional) along a trajectory C (in a space with as many dimensions as $\vec{F}$) is given by a line integral

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\mathbf{T}} ds.$$

In coordinates, in space: $\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ where P, Q, R are functions of x, y, z it becomes $\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz$.

In the plane: $\vec{F} = (M, N)$ where M, N are functions of x, y, it becomes

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\mathbf{T}} ds = \int_C M dx + N dy.$$

(Need to know: geometry so you don't need to compute!)
Evaluation: to evaluate a line integral, we reduce to a single parameter and substitute. E.g. $x = x(t), y = y(t)$ (but could also be in terms of x or y or some angle θ) and express
Special case: if curl $\vec{F} = N_x - M_y = 0$ and $\vec{F}$ is defined and differentiable everywhere then $\vec{F} = \nabla f$ for some potential $f(x, y)$. ($\vec{F}$ is a gradient field) Then

$$f_x = M(x, y) \quad f_y = N(x, y).$$

To find $f(x, y)$: integrate the first equation w.r.t x, take derivatives w.r.t. y and compare with the second equation. We can avoid computing the line integral for work by using the change in potential, via the *Fundamental theorem of Calculus:*

$$\text{work} = \int_C \nabla f \cdot d\vec{r} = f(\text{ end point of } C) - f(\text{start point of } C).$$

Attention: this holds only for work!

Flux in the plane

$\vec{F} = (M, N)$ where M, N are functions of x, y,
The flux of $\vec{F}$ across the plane curve C is by definition by

$$\text{flux} = \int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \int_C M dy - N dx$$

where $\hat{\mathbf{n}}$ = normal vector to C, rotated 90° clockwise from $\hat{\mathbf{T}}$. (picture drawn; explained how the coordinate formula comes from the fact that when we rotate vector (a, b) 90° clockwise, we get $(b, -a)$.
Physical interpretation: if $\vec{F}$ is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time, counting positively what flows towards the right of C, negatively what flows towards the left of C, as seen from the point of view of a point traveling along C. Note: work and flux have different physical interpretations, but they are both line integrals, so they get setup and evaluated the same way.

Green's Theorem

$\vec{F} = (M, N)$ where M, N are functions of x, y
C = closed plane curve that encloses region R counterclockwise

1. for work:
$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F} dA$$
in coordinates:
$$\int_C M dx + N dy = \iint_R (N_x - M_y) dA$$
2. vector form for work:
$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\operatorname{curl} \vec{F}) \hat{\mathbf{k}}\ dA = \iint_R \nabla \times \vec{F}\ dA$$
3. for flux (normal form):
$$\text{flux out of } R = \int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \iint_R (\operatorname{div} \vec{F})\ dA.$$
in coordinates:
$$\int_C M dy - N dx = \iint_R (M_x + N_y) dA$$

Example: we discussed Problem 14 from the study guide.

Flux in space

$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ where P, Q, R are functions of x, y, z
S = surface in space

1. S = parametric surface with parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in R$ some region of the uv-plane.

$$\text{Flux } = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (\Phi_u \times \Phi_v) du dv.$$

2. $S =$ graph of a function $g(x, y)$ with x, y in some region R of the xy-plane.

$$\text{Flux } = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (-g_x, -g_y, 1) dA.$$

3. $S =$ implicit surface given by equation $f(x, y, z) = 0$.

$$\text{Flux } = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (\nabla f) \frac{1}{\nabla f \cdot \hat{\mathbf{k}}} dA,$$

where R is the shadow of S on the xy-plane.

Not covered due to lack of time: surface area, how to use Green's theorem to compute area of a plane region.

MATH 20E Lecture 21 - Friday, May 17, 2013: second midterm