MATH 20E Lecture 22 - Monday, May 20, 2013

Started by discussing Problem 2 on the midterm and FTC.

Stokes' Theorem

This is another example of FTC in action.

Stokes' Theorem: If C is a closed curve in space, and S any surface bounded by C, then

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS}$$

Orientation: compatibility of an orientation of C with an orientation of S (changing orientation changes sign on both sides of Stokes).

Rule: if I walk along C in positive direction, with S to my left, then $\hat{\mathbf{n}}$ is pointing up. (Various examples shown.)

Another formulation (right-hand rule): if thumb points along C (1-D object), index finger towards S (2-D object), then middle finger points along $\hat{\mathbf{n}}$ (3-D object).

(Various examples shown.)

Remark: In Stokes theorem we are free to choose any surface S bounded by C! (e.g. if $C =$ circle, S could be a disk, a hemisphere, a cone, ...)

Example: verify Stokes for $F = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} + y\hat{\mathbf{k}}$, $C =$ unit circle in xy-plane (counterclockwise), $S =$ piece of paraboloid $z = 1 - x^2 - y^2$.

Direct calculation: $x = \cos t, y = \sin t, z = 0$, so

$$\int_C \vec{F} \cdot d\vec{r} = \int_C zdx + xdy + ydz = \int_C xdy = \int_0^{2\pi} \cos^2 tdt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = \pi.$$

By Stokes: $\nabla \times \vec{F} = (1, 1, 1)$, and $\hat{\mathbf{n}}dS = (-g_x, -g_y, 1)dxdy$ for $g(x, y) = 1 - x^2 - y^2$. So $\hat{\mathbf{n}}dS = (2x, 2y, 1)dxdy$ and

$$\iint_S (\nabla\times\vec{F})\cdot\hat{\mathbf{n}}dS = \iint_{\text{unit disk}} (1,1,1)\cdot(2x, 2y, 1)dxdy = \iint (2x+2y+1)dxdy = \iint 1dxdy = \text{ area(disk) } = \pi.$$

($\iint xdxdy = 0$ by symmetry and similarly for y).

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Attention! A special case of Stokes' Theorem is when $S =$ surface in space that has no boundary, i.e. it is closed (e.g. sphere, torus). Then Stokes' Theorem tells us that $\iint_S \nabla \times \vec{F} d\vec{S} = 0$ (the flux of the vector curl across S is 0). This holds for any $\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ where P, Q, R are functions of x, y, z defined and differentiable everywhere on S.

Example 1: Let $\vec{F} = -2xz\hat{\mathbf{i}} + y^2\hat{\mathbf{k}}$. a) Calculate $\nabla \times \vec{F}$. b) Show that $\iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}}dS = 0$ for any portion R of the unit sphere $x^2 + y^2 + z^2 = 1$. (take the normal vector fi pointing outward) c) Show that $\int_C \vec{F} \cdot d\vec{r} = 0$ for any simple closed curve C on the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: a) $\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2xz & 0 & y^2 \end{vmatrix} = (2y, -2x, 0).$

b) On the unit sphere, $\hat{\mathbf{n}} = (x, y, z)$ so $(\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} = 2yx - 2xy = 0$. Therefore $\iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = 0$.

c) By Stokes' Theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS$ where R is the region delimited by C on the unit sphere. Using the result of b), we get $\int_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = 0$.

Example 2: Let C be a simple closed plane curve going counterclockwise around a region R. Let $M = M(x, y)$. Express $\int_C M dx$ as a double integral over R.

Solution: this is an application to Green's theorem. Get $\int_C M dx = \iint_R -M_y dA$.

Example 3: Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying in $z > 1$. Orient S upwards and give its bounding circle, C, lying in $z = 1$ the compatible orientation. a) Parametrize C and use the parametrization to evaluate the line integral

$$I = \int_C xz dx + y dy + y dz.$$

b) Compute the vector curl of the vector field $\vec{F} = xz\hat{\mathbf{i}} + y\hat{\mathbf{j}} + y\hat{\mathbf{k}}$.

c) Write down a flux integral through S which can be computed using the value of I.

Solution: a) $z = 1$ and $x^2 + y^2 + z^2 = 1$, so $x^2 + y^2 = 1$. Therefore C is the circle of radius 1 in the $z = 1$ plane. Compatible orientation: counterclockwise.

Parametrization: $x = \cos t, y = \sin t, z = 1$ Therefore $dx = -\sin t dt, dy = \cos t dt, dz = 0$.

$$I = \int_C xz dx + y dy + y dz = \int_0^{2\pi} (-\cos t \sin t + \cos t \sin t) dt = 0.$$

b)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & y & y \end{vmatrix} = \hat{\mathbf{i}} + x\hat{\mathbf{j}}.$$

c) By Stokes' Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS.$$

$\hat{\mathbf{n}}$ is the normal pointing upwards, so $\hat{\mathbf{n}} = \frac{(x,y,z)}{\sqrt{2}}$ on the upper hemisphere of radius $\sqrt{2}$. Thus

$$I = \int_C \vec{F} \cdot d\vec{r} = \iint_S (1, x, 0) \cdot \frac{(x, y, z)}{\sqrt{2}} dS = \iint_S \frac{x + xy}{\sqrt{2}} dS.$$

MATH 20E Lecture 24 - Friday, May 24, 2013

Divergence Theorem (Gauss-Green Theorem)

This is the 3D analogue of Green's theorem for flux. **Divergence theorem:** If S is a closed surface bounding a region W, with normal pointing outwards, and $\vec{F}$ is a vector field defined and differentiable over all of W, then

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F} dV.}$$

In coordinates, for $\vec{F} = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$:

$$\boxed{\iint_S (P, Q, R) \cdot \hat{\mathbf{n}} dS = \iiint_W (P_x + Q_y + R_z) dV}$$

Example: flux of $\vec{H} = z\hat{\mathbf{k}}$ out of sphere of radius a (seen in Lecture 17): $\operatorname{div} \vec{H} = 0 + 0 + 1 = 1$, so

$$\iint_S \vec{H} \cdot d\vec{S} = \iiint_W 1 dV = \operatorname{vol}(W) = \frac{4\pi a^3}{3}.$$

Physical interpretation: $\operatorname{div} \vec{F}$ = source rate = flux generated per unit volume. Imagine an incompressible fluid flow (i.e. a given mass occupies a fixed volume) with velocity $\vec{F}$, then $\iint_W \operatorname{div} \vec{F} dV = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS$ says that flux through S is the net amount leaving W per unit time = total amount of sources (minus sinks) in W.

Examples: did exercises 3 and 8 from Section 8.4 in the textbook.

Example: take S to be the upper hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Compute the flux of $\vec{F} = 3xy^2\hat{\mathbf{i}} + 3x^2y\hat{\mathbf{j}} + z^3\hat{k}$ upward through S.

Flux = $\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS$. In this case $\hat{\mathbf{n}} = (x, y, z)$ and $\vec{F} \cdot \hat{\mathbf{n}} = 6x^2y^2 + z^3$. So flux = $\iint_{\text{unit circle}} 6x^2y^2 + z^4 dS = \iint_{\text{unit circle}} 6x^2y^2 + (1 - x^2 - y^2)^2 dxdy$. Need to parametrize and deal with powers of trig functions, it gets ugly.

We would like to apply Gauss-Green, but cannot do it directly. Instead take S_1 = unit disk in the xy-plane with normal pointing down. Then $S + S_1$ enclose the upper half-ball W of radius 1 and the divergence theorem says that

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS + \iint_{S_1} \vec{F} \cdot \hat{\mathbf{n}} dS = \iiint_W (\operatorname{div} \vec{F}) dV.$$

To finish next time.