MATH 20E Lecture 25 - Wednesday, May 29, 2013

Example from last time: take S to be the upper hemisphere $x^2+y^2+z^2=1$ with $z \geq 0$. Compute the flux of $\vec{F} = 3xy^2\hat{\mathbf{i}} + 3x^2y\hat{\mathbf{j}} + z^3\hat{k}$ upward through S.

Flux $= \iint_S \vec{F}\cdot\hat{\mathbf{n}}dS$. In this case $\hat{\mathbf{n}} = (x,y,z)$ and $\vec{F}\cdot\hat{\mathbf{n}} = 6x^2y^2+z^3$. So flux $= \iint_{\text{unit circle}} 6x^2y^2 + z^4dS = \iint_{\text{unit circle}} 6x^2y^2 + (1-x^2-y^2)^2dxdy$. Need to parametrize and deal with powers of trig functions, it gets ugly.

We would like to apply Gauss-Green, but cannot do it directly. Instead take S_1 = unit disk in the xy-plane with normal pointing down. Then $S+S_1$ enclose the upper half-ball W of radius 1 and the divergence theorem says that

$$\iint_S \vec{F}\cdot\hat{\mathbf{n}}dS + \iint_{S_1} \vec{F}\cdot\hat{\mathbf{n}}dS = \iiint_W (\operatorname{div}\vec{F})dV.$$

To finish next time. On S_1 the $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ so $\vec{F}\cdot\hat{\mathbf{n}} = -z^3 = 0$ on S_1. So $\iint_{S_1}\vec{F}\cdot\hat{\mathbf{n}}dS = 0$.

Then $\operatorname{div}\vec{F} = 3(x^2+y^2+z^2)$ and

$$\iint_S \vec{F}\cdot\hat{\mathbf{n}}dS = \iiint_W (\operatorname{div}\vec{F})dV = \int_0^{2\pi}\int_0^{pi/2}\int_0^1 3\rho^4\sin\phi d\rho d\phi d\theta = \frac{6\pi}{5}.$$

Conservative vector fields

Example: $\vec{F} = (yz, xz, xy)$. $C: x=t^3, y=t^2, z=t, 0\leq t\leq 1$. Then $dx = 3t^2dt, dy = 2tdt, dz = dt$ and substitute:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C yzdx + xzdy + xydz = \int_0^1 t^3(3t^2dt) + t^4(2tdt) + t^5dt = \int_0^1 6t^5dt = 1.$$

Same $\vec{F}$, curve C' = segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$. In the xy-plane, $z=0 \implies \vec{F} = xy\hat{\mathbf{k}}$, so $\vec{F}\cdot d\vec{r} = 0$, no work on either C_1 or C_2. For the last segment, $x=y=1, dx=dy=0$, so $\vec{F} = (z,z,1)$ and $d\vec{r} = (0,0,dz)$. We get $\int_{C_3}\vec{F}\cdot d\vec{r} = \int_0^1 1dz = 1$.
Both give the same answer because $\vec{F}$ is conservative, in fact $\vec{F} = \nabla(xyz)$.
Recall the fundamental theorem of calculus for line integrals:

$$\int_{P_0}^{P_1} \nabla f\cdot d\vec{r} = f(P_1) - f(P0).$$

Gradient fields

$\vec{F} = (P,Q,R) \stackrel{?}{=} (f_x, f_y, f_z)$ Then $f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$, so $P_y = Q_x, P_z = R_x, Q_z = R_y \iff \nabla\times\vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y) = (0,0,0)$.

Criterion: $\vec{F}$ is a gradient field if and only if $\nabla\times\vec{F} = 0$ and $\vec{F}$ is defined in whole space or "simply connected" region of the space.

Definition: a region W is simply connected if every closed loop C inside W bounds some surface S inside W.

Examples: the complement of the z-axis is not simply connected (shown by considering a loop encircling the z-axis); the complement of the origin is simply connected. A ball is simply connected. A sphere is simply connected. (Pictures drawn)

In the plane: a region R is simply connected if for every closed loop C inside R the interior bounded by C is contained in R. For instance, the plane with the origin removed is not simply connected. Shown using the unit circle. (Picture drawn.)

MATH 20E Lecture 26 - Friday, May 31, 2013

Recall:
Criterion: $\vec{F}$ is a gradient field if and only if $\nabla \times \vec{F} = 0$ and $\vec{F}$ is defined in whole space or simply connected region of the space.

Examples: sphere is simply connected; torus is not (in fact it has two independent loops that dont bound)

Proof of criterion: assume $\vec{F}$ defined in simply connected region W and with $\nabla \times \vec{F} = 0$. Consider two curves C_1 and C_2 with same end points. Then $C = C_1 - C_2$ is a closed curve so bounds some $S \subset W$. Stokes' Theorem tells us that

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS = 0.$$

Thus we get path independence $\implies$ $\vec{F}$ conservative $\implies$ can find potential

$$f(x, y, z) = \int_A^{(x,y,z)} \vec{F} \cdot d\vec{r}.$$

Here A is some point in W.

Example: (a) for which a, b is $\vec{F} = (axy, x^2 + z^3, byz^2?4z^3)$ a gradient field? (b) For the a, b found above, find a potential for $\vec{F}$.

(a) $P_y = ax = 2x = Q_x$ so $a = 2; P_z = 0 = 0 = R_x; Q_z = 3z^2 = bz^2 = R_y$ so $b = 3$.

(b) Systematic method to find a potential: use path-independence
$f(x_1, y_1, z_1) = \int_C F \cdot d\vec{r} = \int_C 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz$ where C is a curve of your choice going from $(0, 0, 0)$ to (x_1, y_1, z_1).

Use a curve that gives an easy computation, e.g. 3 segments parallel to axes. Namely, take $C = C_1 + C_2 + C_3$ where $C_1 =$ segment from $(0, 0, 0)$ to $(x_1, 0, 0); C_2 =$ segment from $(x_1, 0, 0)$ to $(x_1, y_1, 0); C_3 =$ segment from $(x_1, y_1, 0)$ to (x_1, y_1, z_1).

On $C_1 : x = t, y = 0, z = 0, 0 \leq t \leq x_1; dx = dt, dy = dz = 0 \implies \int_{C_1} F \cdot d\vec{r} = \int_{C_1} 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz = \int_{C_1} 0 = 0$.

On $C_2 : x = x_1, y = t, z = 0, 0 \leq t \leq y_1; dx = 0, dy = dt, dz = 0 \implies \int_{C_2} F \cdot d\vec{r} = \int_{C_2} 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz = \int_0^{y_1} x_1^2 dy = x_1^2 y_1$.

On $C_3 : x = x_1, y = y_1, z = t, 0 \leq t \leq z_1; dx = dy = 0, dz = dt \implies \int_{C_3} F \cdot d\vec{r} = \int_{C_3} 2xydx + (x^2 + z^3)dy + (3yz^2 - 4z^3)dz = \int_0^{z_1} (3y_1 t^2 - 4t^3)dt = y_1 z_1^3 - z_1^4$.

So $f(x_1, y_1, z_1) = x_1^2 y_1 + y_1 z_1^3 - z_1^4$.

Check: $f(x, y, z) = x^2 y + yz^3 - z^4 \implies \nabla f = (2xy, x^2 + z^3, 3yz^2 - 4z^3) = \vec{F}$. ☺

Proof of Green's Theorem

Green's Theorem: $\int_C Mdx + Ndy = \iint_R (N_x - M_y)dA$ where C is a closed curve oriented counter-clockwise enclosing region R of the plane.

Proof: two preliminary remarks:

1) the theorem splits into two identities, $\int_C Mdx = -\iint_R M_y dA$ and $\int_C Ndy = \int_R NxdA$.

2) additivity: if theorem is true for R_1 and R_2 then its true for the union $R = R_1 \cup R_2$ (picture drawn): $\int_C = \int_{C_1} + \int_{C_2}$ (the line integrals along inner portions cancel out) and $\iint R = \iint_{R_1} + \iint_{R_2}$.

Main step in the proof: prove $\int_C Mdx = -\iint_R M_y dA$ for "vertically simple" regions: $a < x < b, f_1(x) < y < f_2(x)$. (picture drawn). This is enough because we can divide any region into such pieces and use additivity.

LHS: break C into four sides (C_1 lower, C_2 right vertical segment, C_3 upper, C_4 left vertical segment);

$\int_{C_2} Mdx = \int_{C_4} Mdx = 0$ since $x =$ constant on C_2 and C_4.

On $C_1 : x = x, y = f_1(x), a \le x \le b$ so $\int_{C_1} M(x,y)dx = \int_a^b M(x, f_1(x))dx$

On $C_3 : x = x, y = f_2(x), b \le x \le a$ (because of the orientation) so

$$\int_{C_3} M(x,y)dx = -\int_a^b M(x, f_2(x))dx$$

$$\int_C = \int_{C_1} + \int_{C_3} = \int_a^b (M(x, f_1(x)) - M(x, f_2(x)))\, dx$$

RHS: $\displaystyle\iint_R -M_y dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} M_y dy dx = -\int_a^b (M(x, f_2(x)) - M(x, f_1(x)))\, dx$ (= LHS).

Similarly $\int_C Ndy = \iint_R N_x dA$ by subdividing into horizontally simple pieces. This completes the proof.