MATH 20C – Final Exam Study Guide

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First, let me warn you that this is by no means a complete list of problems, or topics. Just highlights. The first thing you should do when preparing for the exam is to go through your notes, the online notes, the relevant sections of the book and the homework problems. If you still have trouble with some of the topics, take the book (or another calculus book) and solve more problems related to that topic until you *really* understand how and why things work.

Book content covered

12.1, 12.2, 12.3, 12.4, 12.5, 11.1, 13.1, 13.2

11.2 skip "surface area"

13.3 skip "arc length parametrization"

13.5 skip "understanding the acceleration vector"

14.1

14.2 no epsilon or delta

14.3

14.4 skip "(total) differential df"

14.5, 14.6

14.7 skip "least squares method"

14.8 skip "Lagrange multipliers with multiple constraints"

15.1, 15.2, 15.3, 12.7, 15.4

15.5 skip "moment of inertia"

Topics

vectors; operation with vectors (addition, subtraction, multiplication by scalars, dot product, cross product); determinants, area, volume; lines and planes in 3-space (normal vectors); decomposing a vector into a components along specified direction (projection, $\vec{v}_{||}$ and $\vec{v}_{\perp}$); parametric equations

calculus with vectors: limits, differentiation (product rules, chain rule), integration; velocity and acceleration vectors; speed and arc length; tangent line to a trajectory

functions of several variables: graphs, level curves and surfaces, contour maps; limits, partial derivatives, gradient, chain rule (I, II, III), directional derivatives implicit differentiation, tangent planes to surfaces, tangent lines

optimization in several variables: critical points (local min/local max/saddle/degenerate), second derivative test, global min/max, Lagrange multipliers

double integrals (rectangular and polar coordinates); triple integrals (rectangular and cylindrical coordinates); applications: area, volume, mass, average and weighted average of a function, center of mass; pay special attention to how you setup the integration

Concept review

1. What are $\hat{\imath}, \hat{\jmath}, \hat{k}$?
2. Write down the $\mathbf{v} = \langle 25, -3, 6 \rangle$ as a linear combination of $\hat{\imath}, \hat{\jmath}, \hat{k}$. Explain to yourself in words what this vector means in terms of movement.
3. Formula for vector between two points.
4. Vector addition (formula and picture).
5. Vector subtraction (formula and picture).
6. Multiplication by scalar (formula and picture).
7. Write down a linear combination of your favorite 4 vectors in 3-space and write the result both in coordinates and in terms of $\hat{\imath}, \hat{\jmath}, \hat{k}$.
8. What is the definition for the dot product of two vectors?
9. How do you find the cosine of the angle between two vectors?
10. What is the definition for the cross product? (formula and picture)
11. What does the cross product represent geometrically?
12. What is the area of a triangle in space? How about a parallelogram?
13. How do you find the sine of the angle between two vectors?
14. What is the determinant of three vectors?
15. What is the volume of the parellelipiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$?
16. What is the parametric equation for a line through two points?
17. What is the parameteric equation for a line with direction $\mathbf{v}$ and passing through the point P_0?
18. What are the relative positions of two lines in space? Given two lines, how do you determine in which case you are?
19. Find the intersection of the line through $(1, 0, 10)$ with direction $\langle 2, -1, -1 \rangle$ and the line through $(3, -4, 6)$ and $(-5, 2, 1)$.
20. What is the general equation of a plane?
21. What is the normal vector to the plane?
22. How can you find the normal vector the plane passing through three points P_1, P_2, P_3?

23. What is the equation of the plane passing through three points P_1, P_2, P_3?
24. What is the equation of the plane through P with normal vector $\mathbf{v}$?
25. What are the relative positions of a line and a plane? Given a line and a plane, how do you determine in which case you are?
26. Find the intersection of the line $\mathbf{L}(t) = \langle 1+2t, 4, 1+t\rangle$ and the plane $2x+3y-4z=10$.
27. How do you find the intersection of two planes?
28. What are the relative positions of two planes?
29. Find the intersection of $x+y+z=5$ and $2x+3y-4z=10$.
30. How do you find the parametric equation for the intersection of two surfaces?
31. Pick one such parametrization exercise from Section 13.1 in the book and do it.
32. Write down the rule for taking $\lim_{t\to a} \mathbf{r}(t)$.
33. Write down the rule for differentiating $\mathbf{r}(t)$.
34. Write down the rules for integrating $\mathbf{r}(t)$.
35. What is the chain rule for vectors?
36. What is $\frac{d}{dt}(f(t)\mathbf{r}(t))$?
37. What is the rule for differentiating the dot product?
38. What is the rule for differentiating the cross product?
39. What is the velocity vector?
40. What is the acceleration? (vector or scalar? formula?)
41. What is the speed? (vector or scalar? formula?)
42. What is the arc length? (vector or scalar? formula?)
43. What is equation of the tangent line to a parametric curve $\mathbf{r}(t)$ at the point at time t_0?
44. Take the cycloid and compute the velocity, acceleration, speed. Write down an expression for the length of an arch (could involve limit or integration), but don't evaluate. Write down the equation of tangent line at $(\pi/2-1, 1)$.
45. Make sure you know how to draw the level curves and the contour map of a function.
46. Limits of functions of several variables. How do you compute such a limit? How can you tell that a limit does not exist? Can you use the contour map for this?
47. What are the horizontal and vertical slices of the graph of $f(x,y)$?
48. What are the first order partial derivatives of a function of 2 variables? What about a function of 3 variables? What do they represent geometrically? How do you compute them?
49. Chain rule I, II and III (including the chain rule for paths using gradient and velocity).
50. What are the higher order partial derivatives of a function?

51. Compute $\frac{\partial^3 f}{\partial x \partial z^2}$ and $\frac{\partial^3 f}{\partial x \partial y \partial z}$ of $f(x,y,z) = xe^{-yz}$.

52. What is the gradient of a function?

53. Compute the gradient of $f(x,y,z) = xe^{-y^2 z}$.

54. What is the geometric interpretation of the gradient? (Hint: think normal vector.)

55. Write down the approximation formula for a function of 2 variables. And for a function of 3 variables.

56. Write down the equation of the tangent plane to the surface $f(x,y,z) = c$ at the point $P = (x_0, y_0, z_0)$.

57. Write down the equation of the tangent line to the curve $g(x,y) = c$ at the point $Q = (a,b)$.

58. How do you find the equation of the tangent line to a curve that is defined as the intersection of two surfaces? General method.

59. Now find the equation of the tangent line at $P = (2,4,-1)$ to the curve obtained by intersecting $8x^2 - xy^2 + xz + yz - z^3 + 5 = 0$ and $x^3 z + x^2 yz^2 - xy^2 = -8$.

60. What are directional derivatives? How do you compute them? What is their geometric interpretation?

61. Implicit differentiation.

62. Compute $\frac{\partial x}{\partial z}$ at $(2,4,-1)$ when $8x^2 - xy^2 + xz + yz - z^3 + 5 = 0$.

63. What is the Hessian matrix of $f(x,y)$?

64. Compute the Hessian of $f(x,y) = (x+y^2)\cos(x-y)$.

65. How do you find the critical points of a function?

66. Second derivative test.

67. Make sure you can determine the nature of a critical point (min/max/saddle) by looking at the contour map.

68. How do you find the global min/max of a function?

69. How do you find the min/max of $f(x,y,z)$ under the constraint $g(x,y,z) = c$? That is to say, make sure you understand Lagrange multipliers.

70. What does a double/triple integral represent geometrically?

71. More double/triple integrals: make sure you can draw the picture of the region, take slices to set up the iterated integral.

72. Make sure you can set up double integrals in polar coordinates; recall that $dA = r dr d\theta$.

73. Make sure you can set up triple integrals in cylindrical coordinates; recall that $dV = r dz dr d\theta$.

74. Make sure you can set up triple integrals in spherical coordinates; recall that $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.

75. Make sure that you know what the following physical quantities are and how to compute them: area, volume, mass, average and weighted average of a function, center of mass.

76. For evaluation, need to know: usual basic integrals (e.g. $\int \frac{dx}{x}$); integration by substitution (e.g. $\displaystyle\int_0^1 \frac{2t dt}{\sqrt{1+t^2}} = \int_1^2 \frac{du}{2\sqrt{u}}$ by setting $u = 1+t^2$), integration by parts. DO NOT need to know: complicated trigonometric integrals (e.g. $\int \cos^4 \theta d\theta$).

Practice problems

This is a collection of extra problems to help you prepare for the midterm. The list is not complete and these problems do **not** imply anything about the content of the exam. A reasonable exam would consist of problems 1, 6, 11, 12, 13, 15, 17, 19, 21, 22, 28, 30, 34 plus a couple of true/false questions. Try to solve these problems in 180min. But also go through the rest. *Again, this does not mean the exam will have anything to do with problems 1, 6, 11, 12, 13, 15, 17, 19, 21, 22, 28, 30, 34.*

Start by looking over all the homeworks and the problems suggested on the homework webpage.

Section 14.1: 7, 21, 29, 31, 39

Section 14.2: 1, 7, 9, 11

Section 14.3: 9, 13, 21, 41, 43, 47, 61, 65

Section 14.4: 3, 5, 7, 19, 21, 23, 29

Section 14.5: 1, 5, 7, 11, 19, 21, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 49

Section 14.6: 1, 3, 11, 25, 27

Section 14.7: 1, 7, 9, 23, 29, 31, 35

Section 14.8: 1, 5, 7, 9, 41

Section 15.1: 9, 15, 19, 35, 45

Section 15.2: 5, 7, 9, 11, 15, 17, 19, 21, 29, 31, 33, 35, 39, 43, 45, 47

Section 15.3: 1, 3, 11, 15, 17, 21, 23

Section 12.7: 1, 3, 5, 7, 11, 13, 17, 19, 21, 23

Section 15.4: 1, 3, 7, 9, 13, 21, 23, 25, 29

Section 15.5: 1, 3, 11, 13, 23, 27

Chapter 15 review: 15, 31, 35, 37, 41

1. (a) Find the area of the space triangle with vertices $P_0 = (2, 1, 0), P_1 = (1, 0, 1), P_2 = (2, -1, 1)$.
   (b) Find the equation of the plane containing the points P_0, P_1, P_2.
   (c) Find the intersection of this plane with the line parallel to the vector $\vec{v} = \langle 1, 1, 1\rangle$ and passing through the point $S = (-1, 0, 0)$.

2. Let P, Q and R be the points at 1 on the x-axis, 2 on the y-axis and 3 on the z-axis, respectively.
   (a) Express $\overrightarrow{QP}$ and $\overrightarrow{QR}$ in terms of $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$.
   (b) Find the cosine of the angle between $\overrightarrow{QP}$ and $\overrightarrow{QR}$.
   (c) Find the cosine of the angle at R of the triangle in space PQR.

3. Let $P = (1, 1, 1), Q = (0, 3, 1)$ and $R = (0, 1, 4)$.
   (a) Find the area of the triangle in space PQR.
   (b) Find the plane through P, Q and R expressed in the form $ax + by + cz = d$.
   (c) Is the line through $(1, 2, 3)$ and $(2, 2, 0)$ parallel to this plane? Explain why or why not.

4. Find parametric equations for the tangent line to the curve $x = t^2 - 1, y = t^2 + 1, z = t + 1$ at the point $(-1, 1, 1)$.

5. At what angle do the lines $2x + y = 3$ and $3x - y = 4$ intersect?

6. The motion of a point P is given by the position vector $\vec{r}(t) = 3\cos t\hat{\imath} + 3\sin t\hat{\jmath} + t\hat{k}$. Compute the velocity, the acceleration and the speed of P.

7. (a) Find the normal vector $\vec{N}$ to the plane $4x - 3y - 2z = 6$.
   (b) Let $P(t)$ be a point with position vector $\vec{r}(t)$. Express the property that $P(t)$ lies on the plane $4x - 3y - 2z = 6$ in vector notation as an equation involving $\vec{r}$ and the normal vector to the plane.
   (c) By differentiating your answer to (b), show that $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$ is perpendicular to the normal vector to the plane $\vec{N}$.

8. Find $\mathbf{r}(t)$ if $\mathbf{a}(t) = \langle 0, -\cos t, 2\cos t - t\sin t \rangle, \mathbf{v}(\pi) = \hat{\imath} - \pi\hat{k}, \mathbf{r}(\pi) = \pi\hat{\imath} - \hat{\jmath}$.

9. Find the equation of the tangent line at $(0, 3, 0)$ of $\mathbf{r}(t) = \langle t\sin t, 3\cos t, t + \sin t \rangle$. What is the acceleration at the same point?

10. Find the arc length of the curve $\mathbf{r}(t) = \langle t^3, \sqrt{3}t^2, 2t + 8 \rangle$ starting at $t_1 = 0$.

11. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the the surface of the VW is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1, 0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10.

    (a) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. At $t = 0$, the rear bumper is at $(-1, 0)$.
    (b) Compute the speed of the bug, and find where it is largest and smallest. *Hint: It is easier to work with the square of the speed.*

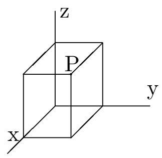
12. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2(x^2 + y^2)}{\sqrt{x^2 + y^2 + 1} - 1}$.

13. Let $f(x, y) = xy - x^4$.

    (a) Draw the traces of f on the three coordinate planes (xy-plane, yz-plane and xz-plane).
    (b) Find the gradient of f at $(1, 1)$.
    (c) Give an approximate formula telling how small changes Δx and Δy produce a small change Δw in the value of $w = f(x, y)$ at the point $(x, y) = (1, 1)$.

14. (a) Find the equation of the tangent plane to the surface $x^3y + z^2 = 3$ at the point $(-1, 1, 2)$.
    (b) Find the point(s) on the surface $3x^2 - 4y^2 = z$ at which the tangent plane is parallel to $3x + 2y + 2z = 7$.

15. A rectangular box is placed in the first octant as shown, with one corner at the origin and the three adjacent faces in the coordinate planes. The opposite point $P = (x, y, z)$ is constrained to lie on the paraboloid $x^2 + y^2 + z = 1$. The goal of this problem is to determine which P gives the box of greatest volume.



(a) Show that the problem leads one to maximize $f(x,y) = xy - x^3y - xy^3$, and write down the equations for the critical points of f.

(b) Find a critical point of f which lies in the first quadrant ($x > 0, y > 0$).

(c) Determine the nature of this critical point by using the second derivative test.

(d) Find the maximum of f in the first quadrant (justify your answer).

16. In the problem above, instead of substituting for z, one could also use Lagrange multipliers to maximize the volume $V = xyz$ with the same constraint $x^2 + y^2 + z = 1$.

(a) Write down the Lagrange multiplier equations for this problem.

(b) Solve the equations (still assuming $x > 0, y > 0$).

17. Let $w = f(u,v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

18. Let $f(x,y) = x^2y^2 - x$.

(a) Find ∇f at $(2,1)$.

(b) Write the equation for the tangent plane to the graph of f at $(2,1,2)$.

(c) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.

(d) Find the directional derivative of f at $(2,1)$ in the direction of $-\hat{\imath} + \hat{\jmath}$.

19. (a) Find the critical points of $w = -3x^2 - 4xy - y^2 - 12y + 16x$ and say what type each critical point is.

(b) Find the point of the first quadrant $x \geq 0, y \geq 0$ at which w is largest. Justify your answer.

20. Let $u = y/x$, $v = x^2 + y^2$, $w = w(u,v)$.

(a) Express the partial derivatives w_x and w_y in terms of x, y, w_u and w_v.

(b) Express $xw_x + yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.

(c) Find $xw_x + yw_y$ in case $w = v^5$.

21. (a) Find the Lagrange multiplier equations for the point of the surface $x^4 + y^4 + z^4 + xy + yz + zx = 6$ at which x is largest. (Do not solve.)

(b) Given that x is largest at the point (x_0, y_0, z_0), find the equation for the tangent plane to the surface at that point.

22. Suppose that $x^2 + y^3 - z^4 = 1$ and $z^3 + zx + xy = 3$. The two surfaces intersect in a curve $\mathcal{C}$. Find the equation of the tangent line to $\mathcal{C}$ at $(1,1,1)$.

23. Evaluate $\lim_{(x,y)\to(0,0)} \left(\frac{1}{2xy} - \frac{1}{xy(xy+2)} \right)$.

24. Draw the contour map for $f(x,y) = x - y^2$ showing at least six level curves.

25. Draw the contour map for $f(x,y) = \frac{1}{1+x^2+y^2}$ showing at least six level curves.

26. Problems 35 and 41 from Section 14.7.

27. Problems 2, 3 from *Preliminary questions,* and Problem 14 from Section 14.8.

28. Set up an iterated integral giving the mass of the solid in space in the shape of the region bounded by the elliptical cylinder $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the sphere of radius 4 centered at $(0,0,2)$ in the first octant $x, y, z \leq 0$, given that the density of the object is $\rho(x,y,z) = y$. Give the integrand and bounds, but DO NOT EVALUATE.

29. Find the volume of the region defined by $4 - x^2 - y^2 \leq z \leq 10 - 4x^2 - 4y^2$.

30. Sketch the region of integration and evaluate by changing to polar coordinates

$$\int_0^2 \int_x^{x\sqrt{3}} x\, dy\, dx.$$

31. (a) Draw a picture of the region of integration of $\displaystyle\int_0^1 \int_x^{2x} dydx$.

    (b) Exchange the order of integration to express the integral in part (a) in terms of integration in the order $dxdy$. Warning: your answer will have two pieces.

32. Set up an iterated integral, in both cylindrical and spherical coordinates, giving the average distance from the origin to the portion of the unit cylinder $x^2 + y^2 \leq 1$ which lies between $z = 0$ and $z = 1$.

33. Set up a triple integral in cylindrical coordinates for the mass of the region of space bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Assume the density $\delta = 3x$.

34. Let S be the surface given by

$$z = (x^2 + y^2 + z^2)^2.$$

    (a) Show that S lies in the upper half space $z \geq 0$.

    (b) Write out the equation for the surface S in spherical coordinates.

    (c) Write down an iterated integral for the volume of the solid W inside the surface S. *Hint: Use part (a).*

    (d) Evaluate the integral.

35. Let D be the domain in the first octant cut off by the plane

$$3x + 2y + z = 1.$$

    Assume R is a solid in the shape of D with density $\delta = z$. Set up an iterated integral in rectangular coordinates for the total mass of R. Evaluate the inner integral only.

36. Let R be a solid in the shape of the first octant of the unit ball. Assume the density is given by $\rho(x,y,z) = y$. Find the z-coordinate of the center of mass of R.

37. Find the volume of the portion of unit ball above the plane $z = 1/\sqrt{2}$.

38. Do more integration problems!