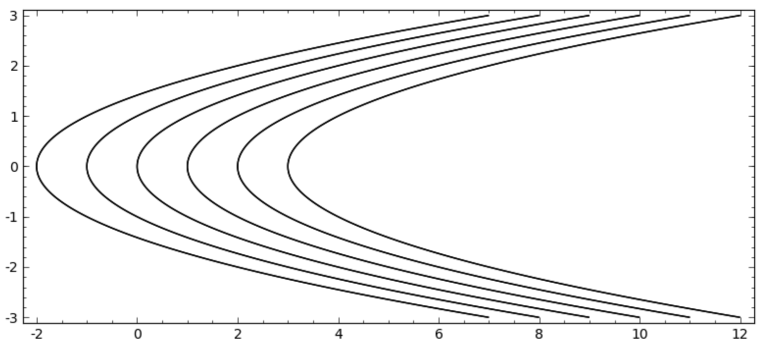
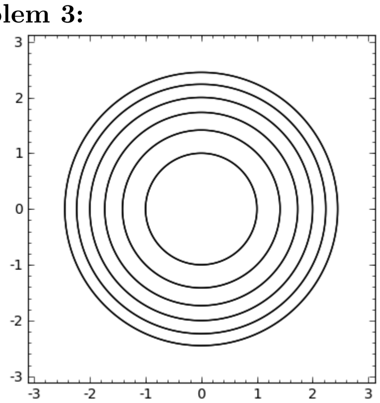
MATH 20C – MIDTERM 2
ANSWERS TO PRACTICE PROBLEMS

ALINA BUCUR

Problem 2:



Problem 3:



Problem 4: 4 (use change of variables $x^2 + y^2 + 1 = t$)

Problem 5: $1/4$

Problem 6: (a) The horizontal trace for $z = 0$: the y-axis and the graph of the function $y = x^3$ in the xy-plane;
the vertical trace when $x = 0$: the y-axis in the yz-plane;
the vertical trace when $y = 0$: the graph of the function $z = -x^4$ in the xz-plane; this is a steeper upside-down parabola.

(b) $\langle -3, 1\rangle$
(c) $\Delta w \approx -3\Delta x + \Delta y$

Problem 7: (a) $3x - y + 4z = 4$
(b) $(-1/4, 1/8, 1/8)$

Problem 8: (a) The volume is $xyz = xy(1 - x^2 - y^2) = xy - x^3y - xy^3$. Critical points: $f_x = y - 3x^2y - y^3 = 0, f_y = x - x^3 - 3xy^2 = 0$.
(b) $(x, y) = (1/2, 1/2)$

(c) local maximum
(d) 1/8

Problem 9: (a) $yz = 2\lambda x, xz = 2\lambda y, xy = \lambda$, and the constraint equation $x^2 + y^2 + z = 1$
(b) $x = 1/2, y = 1/2, z = 1/2$

Problem 10: $\dfrac{\partial w}{\partial x} = yf_u + \dfrac{1}{y} f_v$ and $\dfrac{\partial w}{\partial y} = xf_u - \dfrac{x}{y^2} f_v$.

Problem 11: (a) $\langle 3, 8 \rangle$
(b) $3x + 8y - z = 12$
(c) $f(1.9, 1.1) \approx 2.5$
(d) $\frac{5}{\sqrt{2}}$

Problem 12: (a) $(-20, 34)$ is a saddle point.

(b) There is no critical point in the first quadrant, hence the maximum must be at infinity or on the boundary of the first quadrant.
The boundary is composed of two half-lines:
- $x = 0$ and $y \geq 0$, on which $w = -y^2 - 12y$. It has a maximum ($w = 0$) at $y = 0$.
- $y = 0$ and $x \geq 0$, on which $w = -3x^2 + 16x$ (downwards parabola). It has a maximum when $w_x = -6x + 16 = 0 \ =: x = 8/3$. Hence w has a local maximum at $(8/3, 0)$ and the value is $w = -3(8/3)^2 + 16(8/3) = 64/3 > 0$.

We now check that the maximum of w is not at infinity.
- $y \geq 0$ and $x \to \infty : w \leq -3x^2 + 16x$, which tends to $-\infty$ as $x \to +\infty$.
- $0 \leq x \leq C$ and $y \to \infty : w \leq -y^2 + 16C$, which tends to $-\infty$ as $y \to +\infty$.

We conclude that the maximum of w in the first quadrant is at $(8/3, 0)$.

Problem 13: (a) $w_x - \dfrac{y}{x^2} w_u + 2xw_v$ and $w_y = \dfrac{1}{x} w_u + 2yw_v$.
(b) $2vw_v$
(c) $10v^5$

Problem 14:
(a) $f(x, y, z) = x$ with constraint $g(x, y, z) = x^4 + y^4 + z^4 + xy + yz + zx = 6$. The Lagrange multiplier equation is

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 1 = \lambda(4x^3 + y + z) \\ 0 = \lambda(4y^3 + x + z) \\ 0 = \lambda(4z^3 + x + y). \end{cases}$$

(b) The level surfaces of f and g are tangent at (x_0, y_0, z_0), so they have the same tangent plane. The level surface of f is the plane $x = x_0$; hence this is also the tangent plane to the surface $g = 6$ at (x_0, y_0, z_0).

Second method: at $((x_0, y_0, z_0)$, we have $1 = \lambda g_x, 0 = \lambda g_y, 0 = \lambda g_z$. So $\lambda \neq 0$ and $\langle g_x, g_y, g_z \rangle = \langle 1/\lambda, 0, 0 \rangle$. This vector is therefore perpendicular to the tangent plane to the surface at (x_0, y_0, z_0). The equation of the plane is then $\frac{1}{\lambda}(x - x_0) = 0$, or equivalently $x = x_0$.

Problem 15: $\vec{L}(s) = \langle 1, 1, 1 \rangle + s\langle 16, -16, -4 \rangle$

Problem 18:
(a) The region of integration is the triangle made by the lines $y = x$, $y = 2x$ and $x = 1$. It has vertices $(0, 0)$, $(1, 1)$ and $(1, 2)$.

(b) For $0 \le y \le 1$, have $y/2 \le x \le y$ and for $1 \le y \le 2$, have $y/2 \le x \le 1$. So

$$\int_0^1 \int_x^{2x} dydx = \int_0^1 \int_{y/2}^{y} dxdy + \int_1^2 \int_{y/2}^{1} dxdy.$$

Problem 19: 2