MATH 20C Lecture 1 - Monday, January 6, 2014

Vectors

A vector (notation: $\vec{A}$) has a direction, and a length ($|\vec{A}|$). It is represented by a directed line segment, or arrow. In a coordinate system it's expressed by components. For instance, in space, $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$. (Recall in space x-axis points to the lower-left, y-axis to the right, z-axis up.) It tells me in which direction and how far to move.

Scalar multiplication

Just scale the vector (keep the direction, change the length).

Formula for length

Drew the vector $\langle 1, 2, 3 \rangle$ and asked for its length. Most students got the right answer ($\sqrt{14}$).

Explained how why $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ by reducing to the Pythagorean theorem in the plane. Namely, we first drew a picture showing $\vec{A}$ and its projection on the xy-plane, then derived $\vec{A}$ from the length of the projection and the Pythagorean theorem (applied twice).

Vector addition

Drew a picture of the parallelogram with sides $\vec{A}$ and $\vec{B}$ and showed how the diagonals are $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$. Addition works componentwise and indeed $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ in our earlier example.

Application: Used vector additon to find the components of the vector from point P to point Q. Showed that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$ (where O denotes the origin of the coordinate system).

Observations

1. Two vectors pointing in the same direction are scalar multiples of each other.
2. The sum of three head-to-tail vectors in a triangle is 0.

Lines in space

What kind of object is described by the equation $x + 2y = 5$ in the plane? A line. How about the equation $x + 2y + 3z = 7$ in space? It's a plane. So need some other way to think about lines in space. Think of it as the trajectory of a moving point. Express lines by a parametric equations.
In coordinates. For instance, the line through $Q_0 = (1, 2, 2)$ and $Q_1 = (1, 3, 1)$: moving point $Q(t) = (x, y, z)$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$. Its "velocity" is $\vec{v} = \overrightarrow{Q_0Q_1}$, so $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$. In our example we get $\langle x+1, y-2, z-2 \rangle = t\langle 2, 1, -3 \rangle$, i.e.

$$\begin{cases} x(t) &= -1+2t \\ y(t) &= 2+t \\ z(t) &= 2-3t \end{cases}$$

In vector form. The line through a point P in the direction given by some vector $\vec{v}$ is given by

$$\vec{r}(t) = \overrightarrow{OP} + t\vec{v}$$

where $\vec{r}(t)$ is the position vector of the point $Q(t)$ on the line. Did example with $P = (0, 1, 1)$ and $v = \langle 0, 5, -1 \rangle$.

MATH 20C Lecture 2 - Wednesday, January 8, 2014

Intersection of two lines in space

Attention: always use different variables for the two lines! Find the intersection point P of the lines

$$\vec{r}_1(t) = \langle 0, 1+5t, 1-t \rangle \quad \text{and} \quad \vec{r}_2(s) = \langle 2-s, s+4, s-2 \rangle.$$

Solve system

$$\begin{cases} 2-s &= 0 \\ s+4 &= 1+5t \\ s-2 &= 1-t \end{cases}$$

Get $s = 2, t = 1$ and common point $P = (0, 6, 0)$.

Dot product

Definition $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots$ (a scalar, not a vector)

Geometrically $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$, where θ is the angle between the two vectors.

Explained the result as follows. First, $\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}, \vec{B}$ and $\vec{C} = \vec{A} - \vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta$. On the other hand, we get

$$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

Hence the geometric interpretation is a vector formulation of the law of cosines.

MATH 20C Lecture 3 - Friday, January 10, 2014

Recall that $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots = |\vec{A}||\vec{B}| \cos\theta$.

Today: applications of the dot product

1. **Computing lengths and angles (especially angles)**: $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$.

   For instance, in the triangle in space with vertices at $P = (1,0,0), Q = (0,1,0), R = (0,0,2)$, the angle θ at P:

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1,1,0\rangle\cdot\langle -1,0,2\rangle}{\sqrt{(-1)^2+1^2+0^2}\,\sqrt{(-1)^2+0^2+2^2}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

2. **Relative direction of two vectors**

$$\operatorname{sign}(\vec{A}\cdot\vec{B}) = \begin{cases} >0 & \text{, if } \theta < 90^\circ \text{(acute angle, the two vectors point more or less in the same direction)} \\ =0 & \text{, if } \theta = 90^\circ \Leftrightarrow \vec{A}\perp\vec{B} \\ <0 & \text{, if } \theta > 90^\circ \text{(obtuse angle, vectors point away from each other)} \end{cases}$$

3. **Detecting orthogonality:** it's worth emphasizing that $\vec{A}\perp\vec{B} = 0 \Leftrightarrow \vec{A}\cdot\vec{B} = 0$.

4. **Finding the component of a vector in a given direction** If $\vec{u}$ is a unit vector, the component of a vector A in the direction of $\vec{u}$ has length $|\vec{A}|\cos\theta = |\vec{A}||\vec{u}|\cos\theta = \vec{A}\cdot\vec{u}$ ($\theta =$ the angle between the two vectors). The component itself is a vector called the *projection* of $\vec{A}$ in the direction of $\vec{u}$, namely the vector

$$\vec{A}_{\|} = \mathbf{proj}_{\vec{u}}(\vec{A}) = (\vec{A}\cdot\vec{u})\vec{u}.$$

   The component of $\vec{A}$ in the direction perpendicular to $\vec{u}$ is $\vec{A}_{\perp} = \vec{A} - \vec{A}_{\|}$, so

$$\vec{A} = \vec{A}_{\|} + \vec{A}_{\perp}.$$

   Example: Find the component of $\vec{A} = \langle 1,2,3\rangle$ in the direction of the vector $\vec{v} = \langle 1,1,0\rangle$.

   Step 1 Find the unit vector in the direction of $\vec{v}$:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1,1,0\rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

   Step 2 Find the length of the component:

$$|\vec{A}_{\|}| = \vec{A}\cdot\vec{u} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

   Step 3 Multiply the results from step 1 and step 2:

$$\mathbf{proj}_{\vec{v}}(\vec{A}) = \frac{3}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle = \left\langle \frac{3}{2}, \frac{3}{2}, 0 \right\rangle.$$

   Application: object moving on a frictionless inclined ramp. The physically important quantities are the components of the weight $\vec{F}$ of object (pointing downward) in the direction of the ramp (causes the object to move) and the direction perpendicular to the ramp (creates counter-reaction and keeps the object on the ramp).

5. **Planes**

The plane $x + 2y + 3z = 0$ consists of the points $P = (x, y, z)$ with the property that $\vec{A} \cdot \overrightarrow{OP} = x + 2y + 3z = 0$, where $\vec{A} = \langle 1, 2, 3 \rangle$. This is the same as saying that $\vec{A} \perp \overrightarrow{OP}$.

We say that the vector $\vec{A}$ is a *normal vector* to the plane $x + 2y + 3z = 0$

Another example: find the equation of the plane with normal vector $\langle 2, -1, 1 \rangle$ passing through the point $P = (3, 4, 5)$.

Answer: the equation will be of the form $2x - y + z =$ constant. The value of the constant is determined by the fact that $P = (3, 4, 5)$ has to satisfy the equation of the plane. Get

$$2x - y + z = 5.$$