MATH 20C Lecture 4 - Monday, January 13, 2014

Area

We can decompose the area of a polygon in the plane into a sum of areas of triangles. The area of the triangle with sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}$base×height $= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta = (\frac{1}{2}$area of the parallelogram).

So we need to compute $\sin\theta$. We know how to compute $\cos\theta$. Could do $\sin^2\theta + \cos^2\theta = 1$, but get ugly formula. Instead reduce to complementary angle $\theta' = \frac{\pi}{2} - \theta$ by considering $\vec{A}' = \vec{A}$ rotated by $90° = \frac{\pi}{2}$ counterclockwise (drew a picture).

Then, the area of the parallelogram with sides $\vec{A}, \vec{B}$ is $= |\vec{A}||\vec{B}|\sin\theta = |\vec{A}'||\vec{B}|\cos\theta' = \vec{A}' \cdot \vec{B}$

Continued from last time: If $\vec{A} = \langle a_1, a_2\rangle$ and $\vec{A}' = \vec{A}$ rotated by $90° = \frac{\pi}{2}$ counterclockwise, what are the coordinates of $\vec{A}'$? (showed slide, multiple choice).

Answer: $\langle -a_2, a_1\rangle$ (most students got it right).

So area of the parallelogram with sides $\vec{A}, \vec{B}$ is $= \vec{A}' \cdot \vec{B} = \langle -a_2, a_1\rangle \cdot \langle b_1, b_2\rangle = a_1b_2 - a_2b_1$.

Determinants in the plane

Definition: The determinant of vectors $\vec{A}, \vec{B}$ is $\det\begin{pmatrix}\vec{A}\\ \vec{B}\end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix} = a_1b_2 - a_2b_1$.

Geometrically: $\det\begin{pmatrix}\vec{A}\\ \vec{B}\end{pmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix} = \pm$ area of the parallelogram. (Area is positive, determinant might be negative, so take absolute value.)

Determinants in space

Definition: The determinant of vectors $\vec{A}, \vec{B}, \vec{C}$ is

$$\det\begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}.$$

Geometrically: $\det\begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\end{pmatrix} = \pm$ the volume of the parallelepiped with sides $\vec{A}, \vec{B}, \vec{C}$.

Cross-product

Is defined only for 2 vectors in space. Gives a vector (not a scalar, like dot product).

Definition: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}\hat{\mathbf{k}}$

(the 3×3 determinant is a *symbolic notation*, the actual formula is the expansion).

Geometrically: $\vec{A} \times \vec{B}$ is a vector with

- length: $|\vec{A} \times \vec{B}| =$ area of the parallelogram with sides $\vec{A}, \vec{B}$;

- direction: perpendicular on the plane containing $\vec{A}, \vec{B}$ and pointing in the direction given by the right hand rule.

Right hand rule:

1. extend right hand in direction of $\vec{A}$
2. curl fingers towards direction of $\vec{B}$
3. thumb points in same direction as $\vec{A} \times \vec{B}$

Question Compute $\hat{\mathbf{i}} \times \hat{\mathbf{j}} =$? (multiple choice) Answer: $\hat{\mathbf{k}}$ (most got it right). Checked both by picture and formula.
Another example: $\vec{A} = \langle 5, 2, -7 \rangle$, $\vec{B} = \langle 3, 0, 1 \rangle$. Then

$$\vec{A} \times \vec{B} = \begin{vmatrix} 2 & -7 \\ 0 & 1 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 5 & -7 \\ 3 & 1 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 5 & 2 \\ 3 & 0 \end{vmatrix} \hat{\mathbf{k}} = 2\hat{\mathbf{i}} - 26\hat{\mathbf{j}} - 6\hat{\mathbf{k}} = \langle 2, -26, -6 \rangle.$$

Properties of the cross product:

1. $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$
2. $(2\vec{A}) \times (3\vec{B}) = 6(\vec{A} \times \vec{B})$
3. $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
4. $\vec{A} \times \vec{A} = 0$

MATH 20C Lecture 5 - Wednesday, January 15, 2014

Planes

1. The plane through 3 points, P_1, P_2, P_3.

   A point $P = (x, y, z)$ is in the plane if and only if the volume of the parallelipiped with sides $\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}$ has volume 0 (drew picture). This is the same as saying that

   $$\det \begin{pmatrix} \overrightarrow{P_1P} \\ \overrightarrow{P_1P_2} \\ \overrightarrow{P_1P_3} \end{pmatrix} = 0$$

   Example Take $P_1 = (0, 1, 0), P_2 = (1, 1, 0), P_3 = (1, 0, 0)$. The plane through these 3 points has equation

   $$\det \begin{pmatrix} \langle x, y-1, z \rangle \\ \langle 1, 0, 0 \rangle \\ \langle 1, -1, 0 \rangle \end{pmatrix} = 0$$

   which is to say $z = 0$. This is the xy-plane.

   Note: In general the equation of a plane in space has the form

   $$\boxed{ax + by + cz = d}$$

2. The plane through the origin perpendicular to $\overrightarrow{N} = \langle 1, 5, 10 \rangle$

   Drew a picture. A point $P = (x, y, z)$ is in this plane if and only if $\overrightarrow{OP} \perp \overrightarrow{N}$, which is to say $\overrightarrow{OP} \cdot \overrightarrow{N} = 0$. This means
$$\langle x, y, z \rangle \cdot \langle 1, 5, 10 \rangle = 0,$$
   which gives $x + 5y + 10z = 0$.

3. The plane through $P_0 = (2, 1, -1)$ and perpendicular to $\overrightarrow{N} = \langle 1, 5, 10 \rangle$

   Drew a picture. A point $P = (x, y, z)$ is in this plane if and only if $\overrightarrow{P_0P} \perp \overrightarrow{N}$, which is to say $\overrightarrow{P_0P} \cdot \overrightarrow{N} = 0$. This means
$$\langle x - 2, y - 1, z + 1 \rangle \cdot \langle 1, 5, 10 \rangle = 0,$$
   which gives $x + 5y + 10z = -3$.

   This plane is parallel to the plane in the previous example. In both cases, the coefficients of x, y, z are the components of the vector $\overrightarrow{N}$.

   In the case of $x + 5y + 10z = -3$ one gets the constant -3 by plugging in the coordinates of the point P_0 in the left hand side.

   Definition A vector perpendicular to a plane $\mathcal{P}$ is called a normal vector to that plane. Note that this is implies that all normal vectors to a given plane are proportional.

   So the coefficients of x, y, z are the components of a normal vector to the plane. Conversely, if the equation of the plane is $ax + by + cz = d$, then $\langle a, b, c \rangle$ is a normal vector to it.

Relative positions of lines and planes

Two planes:

are either parallel (if their normal vectors are proportional) or they intersect in a line.

Two lines in space:

Here we have 3 possibilities:

1. parallel (so they are in the same plane): same or opposite direction
2. intersect in a point (again they are in the same plane)
3. skew lines (not in the same plane, but no intersection)

A line and a plane:

To figure it out, take the parametric equation of the line and plug into the equation of the plane. Again, 3 possibilities:

1. the line is parallel the plane (the direction of the line and the normal vector to the plane are perpendicular, but the line and the linear system has no solutions)

2. the line is contained in the plane ((the direction of the line and the normal vector to the plane are perpendicular, and the linear system has infinitely many solutions)

3. the line intersects the plane in a point (the linear system has one solution)

Note: in order to find a normal vector to a plane, just take the cross product of 2 vectors in the plane.

Parametric Equations

In general, parametric equations are a good way to describe arbitrary motions in plane or space. We have already seen an example of this earlier in the course, namely lines in space. It is convenient to think of trajectories in terms of the position vector $\vec{r}(t)$.

1. $\vec{r}(t) = \langle 1+t, 2+t\rangle$ describes a line in the plane in the direction of the vector $\langle 1, 1\rangle$, through the point $(1, 2)$.

2. $\vec{r}(t) = \langle 1+t^3, 2+t^3\rangle$

   Question Does this describe a 1) line? 2) circle? 3) ellipse? (some got it right)

   Answer: line, and in fact the same line as in the previous example.

   To see this, the components are

$$x = 1 + t^3$$
$$y = 2 + t^3$$

   Elliminate the parameter t and get $y = x + 1$.

   Beware! The parametric equation is not unique. That is to say, the same curve in plane or space can be described by many different parametric equations.

3. $\vec{r}(t) = \langle 1+t^2, 2+t^2\rangle$ is only a semiline (part of the same line as in the previous two cases, but only points with coordinates at least $(1, 2)$.

4. $\vec{r}(t) = \langle \cos t, \sin t\rangle$ describes a circle in the plane of radius 1, centered at the origin.

MATH 20C Lecture 6 - Friday, January 17, 2014

Parametric equations - continued

Question: We have seen last time that $\langle \cos t, \sin t\rangle\rangle$ describes the unit circle. What if we take $\vec{r}(t) = \langle \sin t, \cos t\rangle$? Do we still get the unit circle?

The answer is "yes", but the point moves on the trajectory in the opposite direction (clockwise vs. counterclockwise).

Also, $\vec{r}(t) = \langle \cos(2t), \sin(2t)\rangle$ also describes the same circle, but the point moves on it twice as fast.

Question: How to find the parametric equation for a given trajectory?

Example: Find the circle in the plane of radius 5 centered $P = (1, 3)$.

First, the circle of radius 5 centered at the origin has the parametric equation $\langle 5\cos t, 5\sin t\rangle$. In ordered to obtain the desired circle, just translate by $\overrightarrow{OP}$. So we get

$$\vec{r}(t) = \langle 1 + 5\cos t, 3 + 5\sin t\rangle.$$

Question: We have seen last time that $\langle \cos t, \sin t\rangle\rangle$ describes the unit circle. What if we take $\vec{r}(t) = \langle \sin t, \cos t\rangle$? Do we still get the unit circle?

The answer is "yes", but the point moves on the trajectory in the opposite direction (clockwise vs. counterclockwise).

Also, $\vec{r}(t) = \langle \cos(2t), \sin(2t)\rangle$ also describes the same circle, but the point moves on it twice as fast.

Another example: Find a parametric equation for the circle of radius 5 centered at $P = (1, 6, 8)$ lying in a plane parallel to the xz-plane.

We'll follow the same path as before.

Step 1: Write down a parametric equation for the circle centered at the origin, of radius 5, in the xz-plane. Get $\langle 5\cos t, 0, 5\sin t\rangle$.

Step 2 : Translate by the vector $\overrightarrow{OP}$. We get the answer

$$\vec{r}(t) = \langle 1 + 5\cos t, 6, 8 + 5\sin t\rangle.$$

Intersection of surfaces

Another important way to get curves, is by taking the intersection of two surfaces.

We already know an example, namely the intersection of two planes. Going back to the example from the beginning of the lecture, let's find a parametric equation for the intersection of the first 2 planes,

$$\begin{aligned} x + y + 2z &= 7 \\ 2x + y - z &= 4 \end{aligned}$$

We parametrize in terms of $t = x$. That gives

$$\begin{aligned} y + 2z &= 7 - t \\ y - z &= 4 - 2t \end{aligned}$$

Solve and get $y = 5 - \frac{5}{3}t$ and $z = 1 + \frac{1}{3}t$. So our line has direction $\langle 1, -5/3, 1/3\rangle$ and passes through $(0, 5, 1)$.

Another example: Parametrize the intersection of the surfaces

$$\begin{aligned} &x^2 - y^2 = z - 1 \\ &x^2 + y^2 = 4. \end{aligned}$$

We'll do it in two ways.

First way Choose parameter $x = t$. That gives $y^2 = 4 - t^2$ and $z = -3 + 2t^2$. The problem is that when we solve for y we get two different solutions $y = \pm\sqrt{4 - t^2}$. So we need two parametrizations to describe the whole curve:

$$\vec{r}_1(t) = \langle t, \sqrt{4 - t^2}, 2t^2 - 3\rangle$$

and

$$\vec{r}_2(t) = \langle t, -\sqrt{4-t^2}, 2t^2 - 3\rangle.$$

Second way Parametrize the second curve by $x = 2\cos t, y = 2\sin t$ and plug into the first equation. Get $z = 1 + 4\cos^2 t - 4\sin^2 t = 1 + 4\cos(2t)$, so

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 1 + 4\cos(2t)\rangle.$$