MATH 20C Lecture 7 - Wednesday, January 22, 2014

The position vector of a particle moving along a trajectory in plane [or space] is $\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ $[+z(t)\hat{\mathbf{k}}]$.

The velocity vector is $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$ in space or $\vec{v}(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt} \right\rangle$in the plane.

Note: We take derivatives, limits and integrate vectors **componentwise**.

The cycloid: curve traced by a point on a wheel of radius 1 that is rolling on a flat surface at unit speed. It has equation

$$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle.$$

The velocity vector is $\vec{v}(t) = \langle 1 - \cos t, \sin t \rangle$. At $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0 \rangle$.

The *speed* is defined as the magnitude of the velocity vector. In this case,
$|\vec{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$ (smallest at $t = 0, 2\pi, \ldots$, largest at $t = \pi$).

Remark: The speed is $\left|\frac{d\vec{r}}{dt}\right|$ which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Acceleration: $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \vec{r}\,''(t)$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t \rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1 \rangle$ is vertical).

Example: The circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

The velocity vector is $\vec{v}(t) = \langle -\sin t, \cos t \rangle$, and the speed is $|\vec{v}| = 1$ for any t. The acceleration vector is $\vec{a} = \langle -\cos t, -\sin t \rangle$.

Arc length

s = distance travelled along trajectory. Since the rate of change of the distance is the speed, we have $\frac{ds}{dt}$ = speed = $|\vec{v}(t)|$. Can recover length of trajectory by integrating ds/dt. So the length of the curve starting at time t_1 until time t_2 is

$$s = \int_{t_1}^{t_2} |\vec{v}(t)| dt.$$

The distance traveled by a moving point along a curve starting at time t_1 until the current time is

$$s(t) = \int_{t_1}^{t} |\vec{v}(u)| du.$$

Computing such an integral is not always easy... Sometimes it can be done, though. Here are a few examples.

1. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

   Then the velocity is $\vec{v} = \langle -\sin t, \cos t \rangle$ and the speed is $|\vec{v}(t)| = 1$ (contstant). Arc length starting at $t_1 = 0$ and ending at $t_2 = 2\pi$ is

$$s = \int_0^{2\pi} 1 dt = 2\pi$$

2. $\vec{r}(t) = 2t\hat{\mathbf{i}} + (\ln t)\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$

   Then $\vec{v}(t) = 2\hat{\mathbf{i}} + \frac{1}{t}\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$, and the arc length starting at $t_1 = 1$ is

$$s(t) = \int_1^t \sqrt{4 + \frac{1}{u^2} + 4u^2}\, du = \int_1^t \sqrt{\left(2u + \frac{1}{u}\right)^2} du = \int_1^t \left(2u + \frac{1}{u}\right) du = t^2 - 1 + \ln t.$$

Integration

$$\int \vec{r}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle + \vec{c}, \qquad \vec{c} = \langle c_1, c_2, c_3 \rangle$$

Example: $\vec{a}(t) = \hat{\mathbf{k}}, \vec{v}(0) = \hat{\mathbf{i}}, \vec{r}(0) = \hat{\mathbf{j}}$. Find $\vec{r}(t)$.

Since $\vec{a}(t) = \frac{d\vec{v}}{dt}$, that means that $\vec{v}(t) = t\hat{\mathbf{k}} + \vec{c}$. And now $\vec{v}(0) = \hat{\mathbf{i}}$, tells us $\vec{c} = \hat{\mathbf{i}}$. Therefore

$$\vec{v}(t) = t\hat{\mathbf{k}} + \hat{\mathbf{i}}.$$

Repeating the procedure for $\vec{v}(t) = \frac{d\vec{r}}{dt}$, we get

$$\vec{r}(t) = \frac{t^2}{2}\hat{\mathbf{k}} + t\hat{\mathbf{i}} + \hat{\mathbf{j}} = \langle t, 1, t^2/2 \rangle.$$

Tangent lines

The tangent line at time t_0 to the trajectory $\vec{r}(t)$ is the line through the point $P = (x(t_0), y(t_0), z(t_0))$ in the direction given by the velocity vector at that point, $\vec{v}(t_0)$. It has parametric equation

$$\vec{L}(s) = \vec{r}(t_0) + s\vec{v}(t_0).$$

Note: The parameter for the tangent line is different from the parameter of the curve itself.

Take for instance the previous example $\vec{r}(t) = \langle t, 1, t^2/2 \rangle$. Then the tangent line at time $t = 0$ is the line that passes through the endpoint of $\vec{r}(0) = \langle 1, 0, 0 \rangle$ and has direction given by the velocity vector at $t = 0$, $\vec{v}(0) = \langle 0, 1, 0 \rangle$.

Hence

$$\vec{L}(\theta) = \langle 1, 0, 0 \rangle + \theta \langle 0, 1, 0 \rangle.$$

Note that at $\theta = 0$ the line touches the curve, and then it goes away from it.

MATH 20C Lecture 8 - Friday, January 24, 2014

Review

- vectors: length and direction, additions, subtraction, multiplication by scalars
- dot product (**scalar!**): definition, connection with the cosine of the angle between vectors.
- cross product (**vector!**): only in 3D; definition, direction $\perp \vec{A}, \vec{B}$ (and right hand rule), length $=$ area of parallelogram with sides $\vec{A}, \vec{B}$; connection with the sine of the angle between vectors.

- planes: $ax + by + cz = d$
  Then $\langle a, b, c\rangle$ is a normal vector.
  To write down equation of a plane need normal vector and a point in the plane.
- lines: parametric equations
- lines and planes (intersect, parallel, etc)
- parametrize curves and motions
- velocity (**vector!**), acceleration (**vector!**), speed (**scalar!**) of a particle moving along a trajectory $\vec{r}(t)$
- arc length (**scalar or function of time!**) of a trajectory $\vec{r}(t)$
- tangent line to a trajectory
- find the component of a vector in a given direction

We went through the some of the extra problems in the study guide: 2, 3, 7. We also computed arc length in 7.

Example 1 A planet is circling a star counterclockwise describing a circle of radius 5. It makes 2 complete revolutions around the star each year. Parametrize the trajectory of the planet as function of t = time in years.

Since the planet rotates twice around the star in a year, that means that in a year it covers twice the circumference of the circle, that is 4π times the radius. So

$$\vec{r}(t) = \langle 5\cos(4\pi t), 5\sin(4\pi t)\rangle.$$

Example 2 Parametrize the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Take $\frac{x}{a} = \cos t$ and $\frac{y}{b} = \sin t$. We get

$$\vec{r}(t) = \langle a\cos t, b\sin t\rangle.$$