MATH 20C Lecture 12 - Monday, February 3, 2014

Partial derivatives

$f_x = \dfrac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$; same for f_y.

Geometric interpretation: f_x, f_y are slopes of tangent lines of vertical slices of the graph of f (fixing $y = y_0$; fixing $x = x_0$).
How to compute: treat x as variable, y as constant.
Example: $f(x, y) = x^3y + y^2$, then $f_x = 3x^2y, f_y = x^3 + 2y$.
Another example: $g(x, y) = \cos(x^3y + y^2)$.
Use chain rule (version I)

$$\boxed{\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}}$$

Here $F(u) = \cos u$ and $u = f$, so get $\dfrac{\partial g}{\partial x} = -(3x^2y)\sin(x^3y + y^2)$.

Linear approximation

Linear approximation formula:

$$\Delta f \approx f_x \Delta x + f_y \Delta y.$$

Justification: f_x and f_y give slopes of two lines tangent to the graph:

$$L_1 : \begin{cases} y = y_0 \\ z = z_0 + f_x(x_0, y_0)(x - x_0) \end{cases} \quad \text{and } L_2 : \begin{cases} x = x_0 \\ z = z_0 + f_y(x_0, y_0)(y - y_0). \end{cases}$$

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Approximation formula = the graph is close to its tangent plane.

MATH 20C Lecture 13 - Wednesday, February 5, 2014

Recall chain rule I: $g = F(u)$ and $u = u(x, y)$, then $\dfrac{\partial g}{\partial x} = \dfrac{dF}{du}\dfrac{\partial u}{\partial x}$. Used this to compute the partial derivatives of $g(x, y, z) = \ln(x^2 + y^2 - xz)$. Get

$$\frac{\partial g}{\partial x} = \frac{2x - z}{x^2 + y^2 - xz}, \quad \frac{\partial g}{\partial z} = \frac{-x}{x^2 + y^2 - xz}.$$

Higher order partial derivatives

Are computed by taking successive partial derivatives. For instance $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ and so on.

Computed

$$\frac{\partial^2 g}{\partial z \partial x} = \frac{\partial x}{\partial z}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{2x - z}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (2x - z)(-x)}{(x^2 + y^2 - xz)^2}$$

$$\frac{\partial^2 g}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (-x)(2x - z)}{(x^2 + y^2 - xz)^2}$$

Notice that $\frac{\partial^2 g}{\partial z \partial x} = \frac{\partial^2 g}{\partial x \partial z}$. This is no coincidence. In general,

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

Differentials

Recall in single variable calculus: $u = f(x) \implies du = f'(x)dx$.

Example: $u = \arcsin(x) \implies x = \sin u \implies dx = \cos u du \implies \dfrac{du}{dx} = \dfrac{1}{\cos u} = \dfrac{1}{\sqrt{1 - x^2}}$.

Total differential of $f = f(x, y, z)$ is

$$df = f_x dx + f_y dy + f_z dz.$$

This is a new type of object, with its own rules for manipulating it (df is not the same as Δf!) It encodes how variations of f are related to variations of x, y, z. We can use it in two ways:

1. as a placeholder for approximation formulas: $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$.
2. divide by dt to get the **chain rule II**: if $x = x(t), y = y(t), z = z(t)$, then f becomes a function of t and $\dfrac{df}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} + f_z \dfrac{dz}{dt}$.

Example: $w = x^2 y + z$ and $x = t, y = e^t, z = \sin t$. Then $dw = 2xydx + x2dy + dz$. This gives $dw/dt = (2tet)1 + (t^2)et + \cos t$, same as what we obtain by substitution into formula for w and one-variable differentiation.

Can justify the chain rule in 2 ways:

1. $dx = x'(t)dt, dy = y'(t)dt, dz = z'(t)dt$, so substituting we get $dw = f_x dx + f_y dy + f_z dz = f_x x'(t)dt + f_y y'(t)dt + f_z z'(t)dt$, hence dw/dt.
2. (more rigorous):$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$, divide both sides by Δt and take limit as $\Delta t \to 0$.

Application: chain rule with more variables

For example $w = f(x, y)$, $x = x(u, v), y = y(u, v)$. Then we can view f as a function of u and v.
Then

$$dw = f_x dx + f_y dy = f_x(x_u du + x_v dv) + f_y(y_u du + y_v dv) = (f_x x_u + f_y y_u)du + (f_x x_v + f_y y_v)dv.$$

Identifying coefficients of du and dv we get

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}}$$

The idea behind each formula is that changing u causes both x and y to change, at rates $\partial x/\partial u$ and $\partial y/\partial u$. The change in x affects f at the rate of $\partial f/\partial x$, for a total effect of $\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}$. At the same time, the change in y affects f at the rate of $\partial f/\partial y$, for a total effect of $\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$. Finally, the two effects add up to produce the change in f given by the first line in the boxed formula.

Example: polar coordinates.
$x = r\cos\theta, y = r\sin\theta$. Then $\dfrac{df}{dr} = f_x\dfrac{\partial x}{\partial r} + f_y\dfrac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$, and similarly $\dfrac{df}{d\theta}$.

MATH 20C Lecture 14 - Friday, February 7, 2014

Recall the chain rule I for $f(x, y, z)$ and $x = x(t), y = y(t), z = z(t)$:

$$\frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

where $\nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle$ is called the *gradient vector* of $f(x, y, z)$. Using this notation, the chain rule can be re-written as follows. On the path described by $\vec{r}(t) = \langle x(t), y(t) \rangle$, we have

$$\frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt} = \nabla f \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

That is,

$$\boxed{\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \nabla f \cdot \vec{v}}$$

where $\vec{v}$ is the velocity vector.
Note: ∇f is a vector whose value depends on the point (x, y, z) where we evaluate f.
Theorem: ∇f is perpendicular to the level surfaces $f = c$.
Proof: take a curve $\vec{r} = \vec{r}(t)$ contained inside level surface $f = c$. Then velocity $\vec{v} = d\vec{r}/dt$ is in the tangent plane, and by chain rule, $dw/dt = \nabla f \cdot vecv = 0$, so $\vec{v} \perp \nabla f$. This is true for every $\vec{v}$ in the tangent plane.

Example 1: $f(x,y,z) = a_1x + a_2y + a_3z$, then $\nabla f = \langle a_1, a_2, a_3\rangle$. The level surface $f = c$ is $a_1x + a_2y + a_3z = c$. This is a plane with normal vector $\langle a_1, a_2, a_3\rangle = \nabla f$, so ∇f is perpendicular on the plane $f(x,y,z) = c$.
Example 2: $f(x,y) = x^2 + y^2$, then $f = c$ are circles, $\nabla w = \langle 2x, 2y\rangle$ points radially out so $\perp$ circles.
Application: the tangent plane to a surface $f(x,y,z) = c$ at a point P is the plane through P with normal vector $\nabla f(P)$.
Example: tangent plane to $x^2 + y^2 - z^2 = 4$ at $(2,1,1)$: gradient is $\langle 2x, 2y, -2z\rangle = \langle 4, 2, -2\rangle$; tangent plane is $4x + 2y - 2z = 8$. (Here we could also solve for $z = \pm\sqrt{x^2 + y^2 - 4}$ and use linear approximation formula, but in general we can't.)

Directional derivatives

We want to know the rate of change of f as we move (x,y) in an arbitrary direction.

Take a unit vector $\hat{u}$ and look at the cross-section of the graph of f by the vertical plane parallel to $\hat{u}$ and passing through the point (x,y). This is a curve passing through the point $P = (x, y, z = f(x,y))$ and we want to compute the slope the tangent line to this curve at P.

Notice that $\frac{\partial f}{\partial x}$ is the directional derivative in the direction of $\hat{\imath}$ and $\frac{\partial f}{\partial y}$ is the directional derivative in the direction of $\hat{\jmath}$.

Notation: $D_{\hat{u}}f(x_0, y_0)$ denotes the derivative of f in the direction of the unit vector $\hat{u}$ at the point (x_0, y_0).

Shown $f = x^2 + y^2 + 1$, and rotating slices through a point of the graph.

How to compute

Say that $\hat{u} = \langle a, b\rangle$. In order to compute $D_{\hat{u}}f(x_0, y_0)$, look at the straight line trajectory $\vec{r}(s)$ through (x_0, y_0) with velocity $\hat{u}$ given by $x(s) = x_0 + as, y(s) = y_0 + bs$. Then by definition $D_{\hat{u}}f(x_0, y_0) = \dfrac{df}{ds}$. This we can compute by chain rule to be $\dfrac{df}{ds} = \nabla f \cdot \dfrac{d\vec{r}}{ds}$. Hence

$$\boxed{D_{\hat{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}.}$$

Example Compute the directional derivative of $f = x^2 + y^3$ at $P = (2,1)$ in the direction of $\vec{v} = \langle 5, 12\rangle$.

$\nabla f = \langle 2x, 3y^2\rangle$ so $\nabla f(P) = \langle 4, 3\rangle$. The unit vector in the direction of $\vec{v}$ is $\hat{u} = \vec{v}/|\vec{v}| = \langle 5/13, 12/13\rangle$. So $D_{\hat{u}}f(P) = \nabla f(P) \cdot \hat{u} = 56/13$. Therefore f is increasing in the direction of $\vec{v}$.

Geometric interpretation: $D_{\hat{u}}f = \nabla f \cdot \hat{u} = |\nabla f| \cos\theta$. Maximal for $\cos\theta = 1$, when $\hat{u}$ is in direction of ∇f. Hence: direction of ∇f is that of fastest increase of f, and $|\nabla f|$ is the directional derivative in that direction.
It is minimal in the opposite direction.
We have $D_{\hat{u}}f = 0$ when $\hat{u} \perp \nabla f$, i.e. when $\hat{u}$ is tangent to direction of level surface.