MATH 20C Lecture 18 - Wednesday, February 19, 2014

Double integrals

Recall integral in 1-variable calculus: $\int_a^b f(x)dx =$ area below graph $y = f(x)$ over $[a, b]$.

Now: double integral $\displaystyle\iint_R f(x,y)dA =$ volume below graph $z = f(x,y)$ over region R in the xy-plane.

Cut R into small pieces $\Delta A_i \implies$ the volume is approximately $\sum f(x_i, y_i)\Delta A_i$. Limit as $\Delta A \to 0$ gives $\iint f(x,y)dA$. (demo: potato cut into french fries)

How to compute the integral? By taking slices: $S(x) =$ area of the slice by a plane parallel to yz-plane (demo: potato chips); then

$$\text{volume } = \int_{x_{\min}}^{x_{\max}} S(x)dx \quad \text{and for given } x, \quad S(x) = \int_{y_{\min}(x)}^{y_{\max}(x)} f(x,y)dy$$

BEWARE! The limits of integration in y depend on x!
In the inner integral, x is a fixed parameter, y is the integration variable. We get an *iterated integral.*
Example 1 $f(x,y) = 1 - x^2 - y^2$ and $R : 0 \le x \le 1, 0 \le y \le 1$.

$$\int_0^1 \int_0^1 \left(1 - x^2 - y^2\right) dy\, dx$$

How to evaluate?
1) inner integral (x is constant):

$$\int_0^1 \left(1 - x^2 - y^2\right) dy = \left[y - x^2 y - \frac{y^3}{3}\right]_{y=0}^{y=1} = \left(1 - x^2 - \frac{1}{3}\right) - 0 = \frac{2}{3} - x^2.$$

2)outer integral: $\displaystyle\int_0^1 \left(\frac{2}{3} - x^2\right) dx = \left[\frac{2}{3}x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Note: $dA = dy\, dx = dx\, dy$, limit of $\Delta A = \Delta y \Delta x = \Delta x \Delta y$ for small rectangles.

Example 2 Same function over the quarter-disk $R : x^2 + y^2 \le 1, 0 \le x \le 1, 0 \le y \le 1$. (computes volume between xy-plane and paraboloid in the first octant).
How to find the bounds of integration? Fix x constant and look at the slice of R parallel to y-axis. Bounds from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is at $x = 0$, last slice is at $x = 1$. So we get

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \left(1 - x^2 - y^2\right) dx\, dy.$$

Note that the inner bounds depend on the outer variable x; the outer bounds are constants!
1) inner integral (x is constant):

$$\int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy = \left[(1 - x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}} = (1 - x^2)^{3/2} - \frac{(1 - x^2)^{3/2}}{3} = \frac{2}{3}(1 - x^2)^{3/2}.$$

2)outer integral:

$$\int_0^1 \frac{1}{3}(1-x^2)^{3/2}dx = \ldots (\text{trig substitution } x = \sin\theta, \text{ double angle formulas}) \ldots = \frac{\pi}{8}.$$

This is complicated! It will be easier to do it in polar coordinates.

Exchanging the order of integration

$\int_0^1\int_0^2 f(x,y)dx\,dy = \int_0^2\int_0^1 f(x,y)dy\,dx$, since region is a rectangle (drawn picture). In general, more complicated!

Example: $\displaystyle\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y} dy\,dx$ (Inner integral has no formula.)

To exchange order: 1) draw the region (here: $x \le y \le \sqrt{x}$ for $0 \le x \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of y, what are the bounds for x? Here: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1\int_{y^2}^{y} \frac{e^y}{y} dx\,dy = \int_0^1 \frac{e^y}{y}(y - y^2)dy = \int_0^1 e^y(1-y)dy \overset{\text{parts}}{=} [e^y(1-y)]_{y=0}^{y=1} + \int_0^1 e^y dy = e - 2.$$

MATH 20C Lecture 19 - Friday, February 21, 2014

Review topics

- Functions of several variables, contour plots.
- Partial derivatives, gradient; approximation formulas, tangent planes, directional derivatives.
- higher order partial derivatives
- chain rule, change of variables, implicit differentiation
- Min/max problems: critical points, second derivative test, checking boundary. (least squares won't be on the exam)
- Min/max for non-independent variables: Lagrange multipliers
- Double integrals: how to evaluate (iterated integrals) and how to choose/interchange order of integration

Went through problems 5, 7, 8, 14, 18 from the suggested practice problems in the study guide.