MATH 20C Lecture 20 - Monday, February 24, 2014: Second midterm

MATH 20C Lecture 21 - Wednesday, February 26, 2014

Written before lecture, material presented might be slightly different.

Applications of double integrals

Computing volumes *Example:* Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \leq x \leq 2$.

Both surfaces look like parabola-shaped tunnels along the x-axis. They intersect at $1 - y^2 = y^2 - 1 \implies y = \pm 1$. So $z = 0$ and x can be anything, therefore lines parallel to the x-axis. Draw picture, please! Get volume by integrating the difference $z_{\text{top}} - z_{\text{bottom}}$, i.e. take the volume under the top surface and subtract the volume under the bottom surface (same idea as in 1 variable).

$$\begin{aligned} \pm \text{ vol } &= \int_0^2 \int_{-1}^1 \big((1-y^2) - (y^2 - 1)\big)\, dy\, dx = 2\int_0^2 \int_{-1}^1 (1-y^2) dy\, dx \\ &= 2\int_0^2 \left[y - \frac{y^3}{3}\right]_{y=-1}^{y=1} dx = 2\int_0^2 \frac{4}{3} dx = \frac{16}{3}. \end{aligned}$$

Since volume is always positive, our answer is 16/3.

Area of a plane region R is

$$\text{area}(R) = \iint_R 1 dA.$$

Mass the total mass of a flat object in the shape of a region R with density given by $\rho(x, y)$ is

$$\text{Mass} = \iint_R \rho(x, y) dA.$$

Average the average value of a function $f(x, y)$ over the plane region R is

$$\bar{f} = \frac{1}{\text{area}(R)} = \iint_R f(x, y) dA.$$

Weighted average of the function $f(x, y)$ over the plane region R with density $\rho(x, y)$ is

$$\frac{1}{\text{Mass}} \iint_R f(x, y)\rho(x, y) dA.$$

Center of mass of a plate with density $\rho(x,y)$ is the point with coordinates $(\bar{x},\bar{y})$ given by weighted average

$$\bar{x} = \frac{1}{\text{Mass}} \iint_R x\rho(x,y)dA,$$
$$\bar{y} = \frac{1}{\text{Mass}} \iint_R y\rho(x,y)dA.$$

Example: A plate in the shape of the region bounded by $y = x^{-1}$ and $y = 0$ for $1 \leq x \leq 4$ has mass density $\rho(x,y) = y/x$. Calculate the total mass of the plate.

First, draw region. Then set limits of integration.

$$\text{Mass} = \int_1^4 \int_0^{x^{-1}} \frac{y}{x} dy\, dx = \int_1^4 \left[\frac{y^2}{2x}\right]_{y=0}^{y=x^{-1}} dx = \frac{1}{2}\int_1^4 x^{-3} dx = -\frac{1}{4}\left[\frac{1}{x^2}\right]_{x=1}^{x=4} = \frac{15}{64}.$$

For the same region, center of mass has coordinates

$$\bar{x} = \frac{1}{\text{Mass}} \iint_R x\rho(x,y)dA = \frac{64}{15}\int_1^4 \int_0^{x^{-1}} y dy\, dx = \frac{64}{15}\int_1^4 \left[y^2\right]_{y=0}^{y=x^{-1}} dx =$$

$$= \frac{64}{15}\int_1^4 x^{-2} dx = \frac{64}{15}\left[-\frac{1}{x}\right]_{x=1}^{x=4} = \frac{16}{5}$$

and

$$\bar{y} = \frac{1}{\text{Mass}} \iint_R y\rho(x,y)dA = \frac{64}{15}\int_1^4 \int_0^{x^{-1}} \frac{y^2}{x} dy\, dx =$$

$$= \frac{64}{15}\int_1^4 \left[\frac{y^3}{3x}\right]_{y=0}^{y=x^{-1}} dx = \frac{64}{45}\int_1^4 x^{-4} dx = \frac{64}{45}\left[-\frac{1}{3x^3}\right]_{x=1}^{x=4} = \frac{64}{135}\frac{63}{64} = \frac{7}{15}.$$

MATH 20C Lecture 22 - Friday, February 28, 2014

Written before lecture, material presented might be slightly different.

Polar coordinates

Recall: in the plane, $x = r\cos\theta, y = r\sin\theta$ where r is the distance from the origin to the (x,y) point, θ is the angle with the positive x-axis. Drawn picture.
Useful if either integrand or region have a simpler expression in polar coordinates.
Area element: $\Delta A \approx (r\Delta\theta)\Delta r$ (picture drawn of a small element with sides Δr and $r\Delta\theta$). Taking $Deltar, \Delta\theta \to 0$, we get

$$\boxed{dA = r\,dr\,d\theta.}$$

Example (from way back in Lecture 18):

$$\iint_{x^2+y^2\le 1, 0\le x\le 1, 0\le y\le 1} (1-x^2-y^2)\, dxdy = \int_0^{\pi/2}\int_0^1 (1-r^2)r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \frac{\pi}{8}.$$

Once again,

$$\boxed{\iint_R f(x,y)dA = \iint_R f(r,\theta) r\, dr\, d\theta.}$$

In general: when setting up $\iint f r\, dr\, d\theta$, find bounds as usual: given a fixed θ, find initial and final values of r (sweep region by rays).

Example 1 Integrate $xy+y^2$ over the region in plane described in polar coordinates by $1\le r\le 2$, $-\pi/2\le\theta\le\pi/2$.

This is a half annulus. In polar coordinates, $xy+y^2 = r^2\cos\theta\sin\theta + r^2\sin^2\theta$. So we have to compute

$$\int_{-\pi/2}^{\pi/2}\int_1^2 r^2(\cos\theta\sin\theta+\sin^2\theta) r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}(\cos\theta\sin\theta+\sin^2\theta)\left[\frac{r^4}{4}\right]_{r=1}^{r=2} d\theta$$

$$= \frac{7}{4}\int_{-\pi/2}^{\pi/2}(\cos\theta\sin\theta+\sin^2\theta)d\theta = \frac{7}{8}\int_{-\pi/2}^{\pi/2}(\sin(2\theta)+1-\cos(2\theta))\, d\theta$$

$$= \frac{7}{16}\left[-\cos(2\theta)+2\theta-\sin(2\theta)\right]_{\theta=\pi/2}^{\theta=\pi/2} = \frac{7\pi}{16}.$$

Example 2 $\iint_D (x+1)ydA$, where $D: x\ge 0, y\ge 0, x^2+y^2\le 1$.

$x = r\cos\theta, y = r\sin\theta, 0\le r\le 1, 0\le\theta\le\pi/2$ and the integral becomes

$$\int_0^{\pi/2}\int_0^1 (1+r\cos\theta) r\sin\theta r dr d\theta = \int_0^{\pi/2}\int_0^1 (r^2\sin\theta + r^3\sin\theta\cos\theta) dr d\theta =$$

$$= \int_0^{\pi/2}\left(\frac{1}{3}\sin\theta+\frac{1}{4}\sin\theta\cos\theta\right) d\theta = \frac{1}{3}\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{3}+\frac{1}{8} = \frac{11}{24}.$$

Example 3 $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Denote by A our integral. It will be non-negative since the exponential is positive. Then

$$A^2 = A\cdot A = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy.$$

Changing to polar coordinates, this gives $A^2 = \displaystyle\int_0^{2\pi}\int_0^{\infty} re^{-r^2} dr d\theta$.

The inner integral is equal, via the change of variables $u = r^2$, to

$$\frac{1}{2}\int_0^{\infty} e^{-u} du = \frac{1}{2}.$$

Hence $A^2 = \pi$, and $A = \sqrt{\pi}$.