MATH 20C Lecture 23 - Monday, March 3, 2014

Triple Integrals

$$\iiint_R f(x,y,z)\,dV \quad (R \text{ is a solid in space})$$

Note: $\Delta V = \text{area(base)} \cdot \text{height} = \Delta A \Delta z$, so $dV = dA\,dz = dx\,dy\,dz$ or any permutation of the three.

Example 1 $R:$ the region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$. (picture drawn)

The volume of this region is $\iiint_R 1\,dV = \iint_D \left[\int_{x^2+y^2}^{4-x^2-y^2} dz\right] dA$, where D is the shadow in the xy-plane of the region R.

To set up bounds, (1) for fixed (x,y) find bounds for z : here lower limit is $z = x^2 + y^2$, upper limit is $z = 4 - x^2 - y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x,y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x,y) is z on top surface $\geq z$ on bottom surface? Answer: when $4 - x^2 - y^2 \geq x^2 + y^2$, i.e. $x^2 + y^2 \leq 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get

$$\text{vol}(R) = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx.$$

Actual evaluation would be easier using polar coordinates:

$$\text{vol}(R) = \iint_{x^2+y^2\leq 2} (4-2x^2-2y^2)dA = \int_0^{2\pi}\int_0^{\sqrt{2}} (4-2r^2)r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{8}(4-2r^2)^2\right]_{r=0}^{r=\sqrt{2}} d\theta = 4\pi.$$

Example 2 (based on Example 5 from Section 15.3 in the book) Let R be the region in the first octant (i.e. $x \geq 0, y \geq 0, z \geq 0$) bounded by

$$z = 4 - y^2, \quad y = 2x, \quad z = 0, \quad x = 0.$$

Consider $f(x,y,z) = xyz$. Set up the integral $\iiint_R f(x,y,z)\,dV$, in two different ways.

Showed picture of R and its projections on the various coordinate planes.

Setup #1 Integrate first with respect to z, then with respect to y.

The surface $z = 4 - y^2$ intersects the first quadrant of the xy-plane in the line $y = 2$. The projection of the xy-plane is a triangle bounded by the y-axis and the lines $y = 2$ and $y = 2x$. For each point (x,y) the vertical segment above it goes from $z = 0$ to $z = 4 - y^2$. Get

$$\int_0^1 \int_{2x}^2 \int_0^{4-y^2} xyz\,dz\,dy\,dx.$$

Setup #2 Integrate first with respect to x, then with respect to z.

The projection onto the yz-plane is bounded by the y-axis, the z-axis and the parabola $z = 4-y^2$. For each point (y,z) the segment in the direction of the x-axis goes from $x = 0$ to $x = y/2$. Get

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} xyz\,dx\,dz\,dy.$$

Calculate:

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} xyz\,dx\,dz\,dy = \int_0^2 \int_0^{4-y^2} \left[\frac{x^2yz}{2}\right]_{x=0}^{x=y/2} dz\,dy = \frac{1}{8}\int_0^2 \int_0^{4-y^2} y^3z\,dz\,dy = \frac{1}{16}\int_0^2 y^3(4-y^2)^2dy$$

$$= \frac{1}{16}\int_0^2 (16y^3 - 8y^5 + y^7)dy = \frac{1}{16}\left[4y^4 - \frac{4}{3}y^6 + \frac{1}{8}y^8\right]_{y=0}^{y=2} = \frac{1}{16}\left(64 - \frac{256}{3} + 32\right) = 6 - \frac{16}{3} = \frac{2}{3}.$$

The other way,

$$\int_0^1 \int_{2x}^2 \int_0^{4-y^2} xyz\,dz\,dy\,dx = \int_0^1 \int_{2x}^2 \left[\frac{xyz^2}{2}\right]_{z=0}^{z=4-y^2} dy\,dx = \frac{1}{2}\int_0^1 \int_{2x}^2 xy(4-y^2)^2\,dy\,dx =$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x(4-y^2)^3}{6}\right]_{y=2x}^{y=2} = -\frac{1}{12}\int_0^1 x(4-4x^2)^3dx = -\frac{1}{12}\left[\frac{(4-4x^2)^4}{32}\right]_{x=0}^{x=1} = \frac{2^8}{2^7\cdot 3} = \frac{2}{3}.$$

Applications

Mass the total mass of a solid R with density given by $\rho(x,y,z)$ is

$$\text{Mass (R)} = \iiint_R \rho(x,y,z)\,dV.$$

Average the average value of a function $f(x,y,z)$ over the a solid R is

$$\bar{f} = \frac{1}{\text{vol}(R)} = \iiint_R f(x,y,z)\,dV.$$

Weighted average of the function $f(x,y,z)$ over the solid R with density $\rho(x,y,z)$ is

$$\frac{1}{\text{Mass}(R)}\iiint_R f(x,y,z)\rho(x,y,z)\,dV.$$

Center of mass of a solid with density $\rho(x,y,z)$ is the point with coordinates $(\bar{x},\bar{y},\bar{z})$ given by weighted average

$$\bar{x} = \frac{1}{\text{Mass}(R)}\iiint_R x\rho(x,y,z)\,dV,$$

$$\bar{y} = \frac{1}{\text{Mass}(R)}\iiint_R y\rho(x,y,z)\,dV,$$

$$\bar{z} = \frac{1}{\text{Mass}(R)}\iiint_R z\rho(x,y,z)\,dV.$$

Example: Let R be a solid in the shape of the first octant of the unit ball. Assume the density is given by $\rho(x, y, z) = y$. Find the z-coordinate of the center of mass of R.

Solution First drawn picture of R. The unit sphere has equation $x^2 + y^2 + z^2 = 1$. It intersects the xy-plane in the unit circle $x^2 + y^2 = 1$. We want only the parts with $x \geq 0, y \geq 0, z \geq 0$.

$$\begin{aligned}\bar{z} &= \iiint_R z\rho(x,y,z)\,dV = \iint_{\text{quarter unit disk}} \left[\int_0^{\sqrt{1-x^2-y^2}} yz\,dz\right] dA \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} yz\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[y\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dy\,dx = \ldots = \frac{1}{15}.\end{aligned}$$

MATH 20C Lecture 24 - Wednesday, March 5, 2014

Cylindrical coordinates

(r, θ, z) where $x = r\cos\theta, y = r\sin\theta$. (Drawn picture.) Here r measures distance from z-axis, θ measures angle from xz-plane, z is still the height.
Cylinder of radius 8 centered on z-axis is $r = 8$ (drawn); $\theta = \pi/3$ is a vertical half-plane (drawn).
Volume element: $dV = dAdz$; in cylindrical coordinates, $dA = r\,dr\,d\theta$, so

$$\boxed{dV = r\,dr\,d\theta\,dz.}$$

Once again,

$$\boxed{\iiint_R f(x,y,z)dV = \iiint_R f(r,\theta,z) r\,dr\,d\theta\,dz.}$$

Example R : portion of the half-cylinder $x^2 + y^2 \leq 4$, $x \geq 0$ such that $0 \leq z \leq 3y$. Compute the mass of the solid in the shape of R with mass-density given by $\rho(x, y, z) = z^2$.

Again, it's natural to set this up in cylindrical coordinates. The bounds for z are clear: $z_{\text{min}} = 0$ and $z_{\text{max}} = 3y = 3r\sin\theta$. The shadow on the xy-plane is the quarter disk $x^2 + y^2 \leq 1, x \geq 0, y \geq 0$.

$$\begin{aligned}\text{Mass}(R) = \int_0^{\pi/2} \int_0^2 \int_0^{3r\sin\theta} z^2 r\,dz\,dr\,d\theta &= \int_0^{\pi/2} \int_0^2 r\left[\frac{z^3}{3}\right]_{z=0}^{z=3r\sin\theta} dr\,d\theta \\ &= \int_0^{\pi/2} \int_0^2 9r^4 \sin^3\theta\,dr\,d\theta = 9\int_0^{\pi/2} \frac{32}{5}\sin^3\theta\,d\theta.\end{aligned}$$

To evaluate this last integral, write $\sin^3\theta = \sin\theta(1 - \cos^2\theta)$ and use the substitution $u = \cos\theta$. Do not forget to change the bounds of integration!

Example: The volume of the unit ball $\mathbb{B}$ in $\mathbb{R}^3$ can be computed using cylindrical coordinates ($4\pi/3$).

Example: Let $W : x^2 + y^2 \leq 1, 0 \leq z \leq 1 + x^2 + y^2$.

$$\iiint_W (x^2 + y^2 - z)\,dV = \int_0^{2\pi} \int_0^1 \int_0^{1+r^2} (r^2 - z) r dz dr d\theta = \ldots$$

MATH 20C Lecture 25 - Friday, March 7, 2014

Spherical coordinates

Spherical coordinates: (ρ, ϕ, θ).
ρ = rho = distance to origin; $\phi = \varphi$ = phi = angle down from z-axis; θ = same as in cylindrical coordinates. Diagram drawn in space, and picture of 2D slice by vertical plane with z, r coordinates.

Formulas to remember: $z = \rho\cos\phi, r = \rho\sin\phi$ (so $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$).

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}.$$

The equation $\rho = a$ defines the sphere of radius a centered at 0.

On the surface of the sphere, ϕ is similar to latitude, except it is 0 at the north pole, $\pi/2$ on the equator, π at the south pole.
θ is similar to longitude.
$\phi = \pi/4$ is a cone ($z = r = \sqrt{x^2+y^2}$).
$\phi = \pi/2$ is the xy-plane.

Volume element: $dV = \rho^2 \sin\phi d\rho d\phi d\theta$. To understand this formula, first study surface area on sphere of radius a : picture shown of a "rectangle" corresponding to $\Delta\phi, \Delta\theta$ with sides = portion of circle of radius a, of length $a\Delta\phi$, and portion of circle of radius $r = a\sin\phi$, of length $r\Delta\theta = a\sin\phi\Delta\theta$. So $\Delta S \approx a^2\sin\phi\Delta\phi\Delta\theta$, which gives the surface element $dS = a^2\sin\phi d\phi d\theta$.

The volume element follows: for a small "box", $\Delta V = \Delta S \Delta\rho$, so $dV = d\rho dS = \rho^2\sin\phi d\rho d\phi d\theta$.

Example: volume of the unit ball.

$$\iiint_{x^2+y^2+z^2\le 1} 1\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi} 2\, d\theta = \frac{4\pi}{3}.$$

Example: volume of the portion of unit ball above the plane $z = 1/\sqrt{2}$? (picture drawn). This can be set up in cylindrical or spherical coordinates. For fixed ϕ, θ we are slicing our region by rays straight out of the origin; ρ ranges from its value on the plane $z = 1/\sqrt{2}$ to its value on the sphere $\rho = 1$.

Spherical coordinate equation of the plane:

$$z = \rho\cos\phi = 1/\sqrt{2}, \text{ so } \rho = \frac{1}{\sqrt{2}}\sec\phi.$$

The volume is:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi/\sqrt{2}}^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

(Bound for ϕ explained by looking at a slice by vertical plane θ = constant: the edge of the region is at $z = r = 1/\sqrt{2}$). Evaluation: not done. Final answer: $\dfrac{2\pi}{3} - \dfrac{5\pi}{6\sqrt{2}}$.