1. Let $f(x, y) = x\cos(x - 2y) + xy$.

   (a) (5 points) Find ∇f at $(2, 1)$.

$$f_x = \cos(x-2y) + x(-\sin(x-2y)) + y$$

$$f_y = (-2)x(-\sin(x-2y)) + x$$

at $(2,1)$ get $\nabla f(2,1) = (\cos 0 + 2\cdot\sin 0 + 1,\ -2\cdot 2\cdot(-\sin 0) + 2)$

$$\boxed{\nabla f = (1,2)}$$

   (b) (5 points) Write the equation for the tangent plane to the surface $z = f(x, y)$ at the point $(2, 1, 4)$.

$$z - 4 = 1\cdot(x-2) + 2(y-1)$$

$$\boxed{x + 2y - z = 0}$$

   (c) (10 points) Use a linear approximation to find the approximate value of $f(1.9, 1.1)$.

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

$$f(1.9, 1.1) - f(2,1) \approx 1\cdot(1.9 - 2) + 2(1.1 - 1)$$

$$= -0.1 + 0.2 = 0.1$$

$$f(2,1) = 2\cdot\cos 0 + 2\cdot 1 = 2 + 2 = 4$$

$$\boxed{f(1.9, 1.1) \approx 4.1}$$

2. (30 points) Find the area of the ellipse $(2x+5y-3)^2+(3x-7y+8)^2<1$. *Hint: change variables. You can use* **without proof** *the fact that the area of the unit disk is π. It might not be the best idea to try to sketch the ellipse since it can be very time consuming.*

Change of variables

$u = 2x+5y-3$

$v = 3x-7y+8$

region becomes $u^2+v^2<1$: unit disk

$$\text{area ellipse} = \iint_{(2x+5y-3)^2+(3x-7y+8)^2<1} dx\,dy$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & -7 \end{vmatrix} = -14-15 = -29$$

Jacobian

so $du\,dv = |J|\,dx\,dy = 29\,dx\,dy \Rightarrow dx\,dy = \frac{du\,dv}{29}$

$$\text{area ellipse} = \iint_{u^2+v^2<1} 1 \cdot \frac{du\,dv}{29} = \frac{1}{29} \underbrace{\iint_{u^2+v^2<1} du\,dv}_{\text{area unit disk} = \pi}$$

$$\boxed{= \frac{\pi}{29}}$$

3. Let $\vec{F} = \underbrace{(ax^2y^2 + y^3 + 2)}_{M}\hat{\imath} + \underbrace{(4x^3y + bxy^2 + 5)}_{N}\hat{\jmath}$ be a vector field where a and b are constants.

(a) (10 points) Find the values of a and b for which $\vec{F}$ is a gradient field.

Want a, b such that $M_y = N_x$

$M_y = 2ax^2y + 3y^2$
$N_x = 12x^2y + by^2$

$\Rightarrow 2a = 12 \rightarrow \boxed{a = 6}$
$b = 3 \qquad \boxed{b = 3}$

(b) (20 points) For the values of a and b that you found in part (a), find $f(x, y)$ such that $\vec{F} = \nabla f$.

$\vec{F} = (6x^2y^2 + y^3 + 2)\hat{\imath} + (4x^3y + 3xy^2 + 5)\hat{\jmath}$

$\vec{F} = \nabla f = (f_x, f_y) \Rightarrow f_x = 6x^2y^2 + y^3 + 2$
$f_y = 4x^3y + 3xy^2 + 5$

Integrate f_x w.r.t. x and get

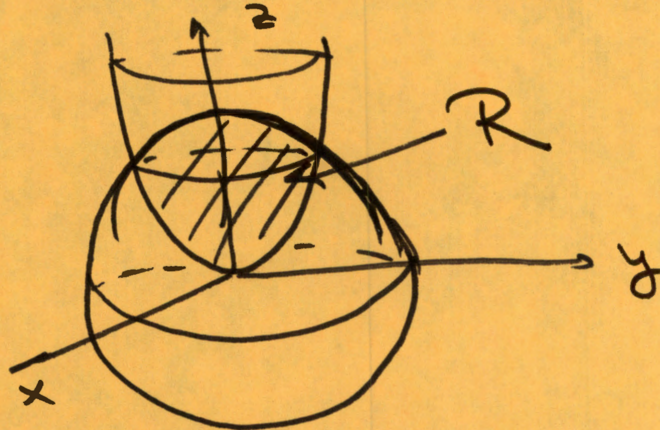
$f = 2x^3y^2 + xy^3 + 2x + g(y) \qquad (*)$

So $f_y = 4x^3y + 3xy^2 + g'(y)$
But $f_y = 4x^3y + 3xy^2 + 5$

$\Rightarrow g'(y) = 5$
$g(y) = 5y + cst$
Plug into (*) and get

$$\boxed{f = 2x^3y^2 + xy^3 + 2x + 5y + cst}$$

4. (a) (5 points) Sketch the region R bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere of radius $\sqrt{2}$ centered at the origin.



(b) (15 points) Set up an iterated triple integral in rectangular coordinates giving the volume of the solid R. Give the integrand and the bounds, but DO NOT EVALUATE.

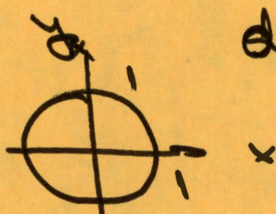
z_{top} : sphree $x^2+y^2+z^2=2 \Rightarrow z=\sqrt{2-x^2-y^2}$

z_{bottom} : paraboloid $z = x^2+y^2$

Shadow on xy-plane : $z = x^2+y^2$, $x^2+y^2+z^2=2$ $\Rightarrow z^2+z=2 \Rightarrow$

$\Rightarrow z = \frac{-1\pm\sqrt{9}}{2} = 1$

disk of radius 1 : $x^2+y^2\le 1$



$$\text{volume} = \iiint_R dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\, dy\, dx$$