MATH 20E Lecture 1 - Thursday, September 26, 2013

Disclaimer: The first few weeks will be mostly review of material from MATH 20C. We still have to go through it, because some of it was skipped in the various versions of 20C taught in the past few years. The aim of these first lectures is **not** to re-teach the concepts of multivariable calculus, but to refresh your memory and make sure everyone is on the same page with notation, etc...

Vectors

Denoted $\vec{v}$ or $\vec{A}$.
A vector has direction, and length/norm ($\|\vec{A}\|$). It is represented by a directed line segment. Drawn vectors $(2,1)$ in plane and $(2,1,2)$ in space. Clicker questions (no credit today!): what is the length of $(2,1)$? What is the length of $(2,1,2)$? Most people got both right.
Formula for the norm: $\|(a_1,a_2,a_3,\dots)\| = \sqrt{a_1^2+a_2^2+a_3^2+\dots}$ (it is a scalar = number)

Attention: you should always know if you are dealing with a vector or a scalar!

Vector addition: parallelogram law

Dot product: $\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + \dots$ is a scalar. Geometrically, $\vec{A}\cdot\vec{B} = \|\vec{A}\|\|\vec{B}\|\cos\theta$.
Note that $\vec{A}\perp\vec{B} \iff \vec{A}\cdot\vec{B}=0$.

In general,

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\|\vec{B}\|}$$

Example: triangle in space with vertices $P=(1,0,0), Q=(0,1,0), R=(0,0,2)$ (picture drawn). Find angle at P.

$$\cos\theta = \frac{\vec{PQ}\cdot\vec{PR}}{\|\vec{PQ}\|\|\vec{PR}\|} = \frac{(-1,1,0)\cdot(-1,0,2)}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}} \implies \theta\approx 71.5^\circ$$

Determinants

in 2D:

$$\det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \pm \text{ area parallelogram with sides } \vec{A},\vec{B}.$$

in 3D:

$$\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \pm\text{vol parallelipiped with sides } \vec{A},\vec{B},\vec{C}.$$

Cross-product: (only for 2 vectors in space); gives a vector, not a scalar (unlike dot-product).
Geometrically: $\vec{A}\times\vec{B}$ has length $\|\vec{A}\times\vec{B}\|$ = area of space parallelogram with sides $\vec{A},\vec{B}$;
direction = normal to the plane containing $\vec{A}$ and $\vec{B}$.
How to decide between the two perpendicular directions = right-hand rule.

1) extend right hand in direction of $\vec{A}$; 2) curl fingers towards direction of $\vec{B}$; 3) thumb points in same direction as $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\mathbf{k}}.$$

(the 3×3 determinant is a *symbolic notation*, the actual formula is the expansion)

Example: $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$ (clicker question, checked both by geometric description and by calculation).

Equations of planes in space

- plane with normal vector $\vec{N} = (3, 5, 2)$ passing through point $P = (-1, 7, 9)$ has equation $\vec{N} \cdot (x+1, y-7, z-9) = 0$. Becomes $3x + 5y + 2z = 50$.
- plane through the points $P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 2)$ from the example before has normal vector $\vec{N} = \vec{PQ} \times \vec{PR} = (2, 2, 1)$. The equation of the plane is $2x + 2y + z = 2$.
- Consider the plane $2x + 5y - z = 18$. What is a normal vector to this plane? Answer: $(2, 5, -1)$.