MATH 20E Lecture 2 - Tuesday, October 1, 2013

More notions from MATH 20C.

Partial derivatives

$f_x = \dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$; same for f_y.

Geometric interpretation: f_x, f_y are slopes of tangent lines of vertical slices of the graph of f (fixing $y = y_0$; fixing $x = x_0$).
How to compute: treat x as variable, y as constant.

In vector notation, for a function of n variables

$$\frac{\partial f}{\partial x_j}(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{e}_j)}{h}$$

where

$$e_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0).$$

Examples: Example: $f(x, y) = x^3 y + y^2$. Then $\frac{\partial f}{\partial x} = 3x^2 y$ (treat y as a constant, x as a variable) and $\frac{\partial f}{\partial y} = x^3 + 2y$.

We can package the partial derivatives into the *gradient vector* $\nabla f(\vec{a}) = \left(\dfrac{\partial f}{\partial x_1}(\vec{a}), \quad \dots \quad , \dfrac{\partial f}{\partial x_n}(\vec{a}) \right)$.

Example: $g(x, y, z) = \ln(x^2 + y^2 - xz)$. Then

$$\frac{\partial g}{\partial x} = \frac{2x - z}{x^2 + y^2 - xz}, \quad \frac{\partial g}{\partial y} = \frac{2y}{x^2 + y^2 - xz}, \quad \frac{\partial g}{\partial z} = \frac{-x}{x^2 + y^2 - xz}.$$

On clicker: $\nabla g(1, 1, 1) = (1, 2, -1)$; $\nabla g(1, 1, 2)$ does not exist (cannot plug in); $\nabla g(1, 1, 3)$ does not exist either, since the function is not defined at that point.

We can also take higher order partial derivatives. For instance,

$$\frac{\partial^2 g}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2 - xz}\right) = \frac{(-1)(x^2 + y^2 - xz) - (-x)(2x - z)}{(x^2 + y^2 - xz)^2}.$$

Linear approximation

$$z = f(x, y)$$

Linear approximation formula: $\Delta f \approx f_x \Delta x + f_y \Delta y$.

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

where $z_0 = f(x_0, y_0)$.
Approximation formula = the graph is close to its tangent plane.

Example: $z=\cos x+e^{1-y^2}$ at $(x_0,y_0)=\left(\frac{\pi}{2},1\right)$. Then $z_0=\cos\frac{\pi}{2}+e^0=1$ and $\frac{\partial z}{\partial x}=-\sin x$, $\frac{\partial z}{\partial y}=-2ye^{1-y^2}$. Thus $a=\frac{\partial z}{\partial x}(x_0,y_0)=\sin\frac{\pi}{2}=-1$ and $b=\frac{\partial z}{\partial y}(x_0,y_0)=-2$. The tangent plane has equation

$$z-1=-1\left(x-\frac{\pi}{2}\right)-2(y-1)\iff x+2y+z=3+\frac{\pi}{2}.$$

Question (clicker): is $f\left(\frac{\pi}{2}+\frac{\pi}{100},0.9\right)$ bigger or smaller than 1?
Answer: linear approximation says that $\Delta f\approx f_x\Delta x+f_y\Delta y$. Plug in our values and see

$$f\left(\frac{\pi}{2}+\frac{\pi}{100},0.9\right)-f\left(\frac{\pi}{2},1\right)\approx(-1)\frac{\pi}{100}+(-2)\cdot 0.1>0\implies f\left(\frac{\pi}{2}+\frac{\pi}{100},0.9\right)-1>0.$$

The differential

The *Jacobian matrix* of $f=(f_1,\dots,f_m)$ of n variables that takes values in $\mathbb{R}^m$ is also called the *differential (derivative)* of f. At a point $\vec{a}=(a_1,\dots,a_n)$ is given by

$$T=Df(\vec{a})=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1}(\vec{a}) & \dots & \dfrac{\partial f_1}{\partial x_n}(\vec{a})\\ \vdots & \ddots & \vdots\\ \dfrac{\partial f_m}{\partial x_1}(\vec{a}) & \dots & \dfrac{\partial f_m}{\partial x_n}(\vec{a})\end{pmatrix}$$

Note that it is an $m\times n$ matrix.
Example: $f(x,y,z)=(\sin(xyz),x^2+y^2-z)$ and $\vec{a}=(1,5,0)$. Then

$$Df=\begin{pmatrix}yz\cos(xyz) & xz\cos(xyz) & xy\cos(xyz)\\ 2x & 2y & -1\end{pmatrix}$$

and

$$Df(\vec{a})=\begin{pmatrix}0 & 0 & 5\\ 2 & 10 & -1\end{pmatrix}$$

Particular case For a function $f:\mathbb{R}^n\to\mathbb{R}$ the differential, which is a $1\times n$ matrix, can be identified with the gradient vector

$$\nabla f(\vec{a})=\left(\frac{\partial f}{\partial x_1}(\vec{a}),\quad\dots\quad,\frac{\partial f}{\partial x_n}(\vec{a})\right)$$

Properties:

- $f:\mathbb{R}^n\to\mathbb{R}^m$, c real number. The differential of $h=cf$ is $Dh(\vec{a})=cDf(\vec{a})$.
- $f,g\mathbb{R}^n\to\mathbb{R}^m$. The differential of $h=f+g$ is $Dh(\vec{a})=Df(\vec{a})+Dg(\vec{a})$.
- (product rule) $f,g:\mathbb{R}^n\to\mathbb{R}$. The differential of $h=fg$ is $Dh(\vec{a})=g(\vec{a})Df(\vec{a})+f(\vec{a})Dg(\vec{a})$.
- (quotient rule) $f,g:\mathbb{R}^n\to\mathbb{R}$. The differential of $h=\frac{f}{g}$ is $Dh(\vec{a})=\dfrac{g(\vec{a})Df(\vec{a})-f(\vec{a})Dg(\vec{a})}{(g(\vec{a}))^2}$.

Chain rule

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$ and set $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$. Then, for $\vec{a} \in \mathbb{R}^n$ and $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$ we have

$$\boxed{D(g \circ f)(\vec{a}) = Dg(\vec{b})\, Df(\vec{a})} \text{ (matrix multiplication)}$$

Special cases:

- $g = F(u)$ and $u = u(x, y, z)$. Then

$$\frac{\partial g}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}$$

  We already used this a couple of times earlier in lecture, e.g. when we computed ∇g for $g(x, y, z) = \ln(x^2 + y^2 - xz)$.

- $\vec{c} : \mathbb{R} \to \mathbb{R}^3, c(t) = (x(t), y(t), z(t))$ path and $f : \mathbb{R}^3 \to \mathbb{R}$. Then the derivative of $h(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$ is

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = (\nabla f(\vec{c}(t))) \cdot \left(\vec{c}'(t)\right).$$

  Example: $\vec{c}(t) = (t, t^2, t^3), f(x, y, z) = x^2 + y^2 - \cos z$. Chain rule gives

$$\frac{dh}{dt} = (2x)1 + (2y)(2t) + (\sin z)(3t^2) = 2t + 4t^3 + 3t^2 \sin(t^3)$$

  where $h(t) = f(\vec{c}(t)) = t^2 + t^4 - \cos(t^3)$.

- $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}$ $f(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$ and the composition $h(x, y, z) = g \circ f = g\left(u(x, y, z), v(x, y, z), w(x, y, z)\right).$

$$\nabla h = \nabla g\, Df.$$

  i.e.

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x}$$

  and so on.

Taylor's formula for $n = 1$: linear approximation. For $n \geq 2$: left to you as reading assignment.

MATH 20E Lecture 3 - Thursday, October 3, 2013

Double integrals

$$\iint_R f(x, y) dA, \quad dA = dxdy = dydx$$

We compute by reducing to an iterated integral

$$\iint_R f(x,y)dA = \int_{y_{\min}}^{y_{\max}} S(y)dy, \quad \text{where } S(y) = \int_{x_{\min}(y)}^{x_{\max}(y)} f(x,y)dx \text{ for each } y$$

Example 1 $f(x,y) = 1 - x^2 - y^2$ and $R : 0 \le x \le 1, 0 \le y \le 1$.

$$\int_0^1 \int_0^1 \left(1 - x^2 - y^2\right) dx\, dy$$

How to evaluate?
1) inner integral (x is constant):

$$\int_0^1 \left(1 - x^2 - y^2\right) dy = \left[y - x^2 y - \frac{y^3}{3}\right]_{y=0}^{y=1} = \left(1 - x^2 - \frac{1}{3}\right) - 0 = \frac{2}{3} - x^2.$$

2)outer integral: $\displaystyle\int_0^1 \left(\frac{2}{3} - x^2\right) dx = \left[\frac{2}{3}x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Example 2 Same function over the quarter-disk $R : x^2 + y^2 \le 1, 0 \le x \le 1, 0 \le y \le 1$.
How to find the bounds of integration? Fix x constant and look at the slice of R parallel to y-axis. Bounds from $y = 0$ to $y = \sqrt{1-x^2}$ in the inner integral. For the outer integral: first slice is at $x = 0$, last slice is at $x = 1$. So we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy\, dx.$$

Note that the inner bounds depend on the outer variable x; the outer bounds are constants!
1) inner integral (x is constant):

$$\int_0^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy = \left[(1-x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}} = (1-x^2)^{3/2} - \frac{(1-x^2)^{3/2}}{3} = \frac{2}{3}(1-x^2)^{3/2}.$$

2)outer integral:

$$\int_0^1 \frac{2}{3}(1-x^2)^{3/2} dx = \ldots (\text{trig substitution } x = \sin\theta, \text{ double angle formulas}) \ldots = \frac{\pi}{8}.$$

This is complicated! It will be easier to do it in polar coordinates.

Example 3 $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \frac{e^x}{x} dx\, dy$ (Inner integral has no formula.)
To exchange order: 1) draw the region (here: $y \le x \le \sqrt{y}$ for $0 \le y \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of x, what are the bounds for y? Picture: left border is $y = x$, right is $x^2 = y$; first slice is $x = 0$, last slice is $x = 1$, so we get

$$\int_0^1 \int_{x^2}^x \frac{e^x}{x} dy\, dx = \int_0^1 \frac{e^x}{x}(x - x^2)dx = \int_0^1 e^x(1-x)dx \overset{\text{parts}}{=} \left[e^x(1-x)\right]_{x=0}^{x=1} + \int_0^1 e^x dx = e - 2.$$

Example 4 Find the volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \le x \le 2$.

Both surfaces look like parabola-shaped tunnels along the x-axis. They intersect at $1 - y^2 = y^2 - 1 \implies y = \pm 1$. So $z = 0$ and x can be anything, therefore lines parallel to the x-axis (picture drawn). Get volume by integrating the difference $z_{\text{top}} - z_{\text{bottom}}$, i.e. take the volume under the top surface and subtract the volume under the bottom surface (same idea as in 1 variable).

$$\begin{aligned}
\text{vol} &= \int_0^2 \int_{-1}^1 \big((1 - y^2) - (y^2 - 1)\big)\, dy\, dx = 2 \int_0^2 \int_{-1}^1 (1 - y^2) dy\, dx \\
&= 2 \int_0^2 \left[y - \frac{y^3}{3} \right]_{y=-1}^{y=1} dx = 2 \int_0^2 \frac{4}{3} dx = \frac{16}{3}.
\end{aligned}$$

Example 5 Did the interchange in order of integration from Example 2, page 291.