MATH 20E Lecture 19 - Tuesday, December 3, 2013

Review for final exam - part I

vectors: dot product ($\vec{v} \cdot \vec{w} = \sum v_i w_i = \|\vec{v}\|\|\vec{w}\| \cos\theta$), cross product ($\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin\theta = areaparalelogram$

functions of several variables

$f : \mathbb{R}^n \to \mathbb{R}$ $f(x, y, z, \ldots)$

partial derivatives: $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z} \ldots$

gradient vector: $\nabla f = (f_x, f_y, f_z, \ldots)$

one example of chain rule:

- if $g = F(u)$ and $u = u(x, y, z)$ then $\frac{\partial g}{\partial x} = \frac{dF}{du} \frac{\partial u}{\partial x}$

Example: Let $w = f(u, v)$, where $u = xy$ and $v = x/y$. Using the chain rule, express $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ in terms of x, y, f_u and f_v.

The chain rule says that $\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y f_u + \dfrac{1}{y} f_v$ and $\dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x f_u - \dfrac{x}{y^2} f_v$.

More generally, the Jacobian matrix of $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ of n variables that takes values in $\mathbb{R}^m$ is given by

$$T = Df(\vec{a}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \dfrac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \dfrac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix}.$$

chain rule: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$ and set $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$. Then, for $\vec{a} \in \mathbb{R}^n$ and $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$ we have $D(g \circ f)(\vec{a}) = Dg(\vec{b})\, Df(\vec{a})$ (matrix multiplication)

Special cases:

- $\vec{c} : \mathbb{R} \to \mathbb{R}^3, c(t) = (x(t), y(t), z(t))$ path and $f : \mathbb{R}^3 \to R$. Then the derivative of $h(t) = f(\vec{c}(t)) = f(x(t), y(t), z(t))$ is

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = (\nabla f(\vec{c}(t))) \cdot \left(\vec{c}'(t)\right).$$

- $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}$ $f(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$, then $h(x, y, z) = g \circ f = g\left(u(x, y, z), v(x, y, z), w(x, y, z)\right)$.

linear approximation formula for $f(x, y, z, \ldots) : \Delta f \approx f_x \Delta x + f_y \Delta y + \ldots$

Did problem 1 from the study guide.

tangent planes to surfaces

- S is $z = f(x, y)$ the graph of $f(x, y)$

  Tangent plane at (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$ ihas equation $f_x(x - x_0) + f_y(y - y_0) = z - z_0$.

  In the example above, $f(x, y) = xy - x^4$ and the tangent plane at $(1, 1)$ is
  $z - f(1, 1) = -3(x - 1) + (y - 1)$, i.e. $z + 3x - y = 2$.

- S is the level surface $g(x,y,z) = 0$, then the tangent plane at (x_0,y_0,z_0) has equation $\nabla g(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$.

  Example: tangent plane to $S: x^2+3y^2z-z^4 = -26$ at the point $(1,3,2)$ The gradient of $g(x,y,z) = x^2+3y^2z-z^4+26$ is $\nabla g = (2x, 6yz, 3y^2-4z^3) = (2,36,-5)$. The tangent plane is given by

$$2(x-1)+36(y-3)-5(z-2) = 0 \iff 2x+36y-5z = 100.$$

- S is parametrized by $\Phi(u,v)$
  normal vector: $\Phi_u \times \Phi_v$

Did problem 5 from the study guide.

double integrals: draw the region!
setup: need bounds of integration, then evaluate first inner integral and then outer.

$$\iint_R f(x,y)dA = \int_{x_{\min}}^{x_{\max}} \int_{y_{\text{bottom}}(x)}^{y_{\text{top}}(x)} f(x,y)dydx$$

polar coordinates: $x = r\cos\theta, y = r\sin\theta \implies dA = rdrd\theta$
Did problem 7 from the study guide.
general change of variables: $x = x(u,v), y = y(u,v) \implies dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$ (absolute value!)

Example: Find the area of the ellipse $x^2/4 + y^2/9 \le 1$.
The area is given by $\iint_{x^2/4+y^2/9\le 1} 1dxdy$
Change $x = 2u, y = 3v$ and get

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 \implies dxdy = |6|dudv = 6dudv.$$

Therefore the area of the ellipse is

$$\iint_{x^2/4+y^2/9\le 1} 1dxdy = \iint_{u^2+v^2\le 1} 6dudv = 6 \cdot \text{area(unit disk)} = 6\pi.$$

triple integrals: setup: need bounds of integration then evaluate innermost integral and get a double integral; now do the double integral

$$\iiint_W f(x,y,z)dV = \iint_{\text{shadow in the } xy\text{-plane}} \left[\int_{z_{\text{bottom}}(x,y)}^{z_{\text{top}}(x,y)} f(x,y,z)dz\right] dA$$

Did problem 6 from the study guide.
rectangular coordinates: $dV = dxdydz$
cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z \implies dV = dzrdrd\theta$
spherical coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi \implies dV = \rho^2\sin\phi d\rho d\phi d\theta$
general change of variables: $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w) \implies dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$ (absolute value again!)

MATH 20E Lecture 20 - Thursday, December 5, 2013

Review for final exam - part II

vector fields: recall flow lines, how to sketch vector fields

work and line integrals: work $= \int_C \vec{F} \cdot d\vec{r}$ where C is a curve in plane, space, etc...

in 2D: $\vec{F} = (M, N) \implies \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$ (to evaluate: express everything in terms of a single parameter)

in 3D: $\vec{F} = (P, Q, R) \implies \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz$ (to evaluate: express everything in terms of a single parameter)

gradient fields and path independence:
If $\vec{F}$ is defined in a simply connected region (in plane or space) and $\nabla \times \vec{F} = 0$, then $\vec{F}$ is a gradient fields, i.e. $\vec{F} = \nabla g$ for some function $g(x, y)$ or $g(x, y, z)$.
To find potential: 2 methods
A. compute a line integral, e.g. $(0, 0)$ to $(x_1, 0)$ to (x_1, y_1)
B. antiderivatives
For gradient fields, work is given by the Fundamental Theorem of Calculus

$$\int_C \nabla g \cdot d\vec{r} = g(\text{end point}) - g(\text{start point}).$$

flux in plane: flux of $\vec{F} = (M, N)$ across a curve C in the plane is given by

$$\text{flux} = \int_C \vec{F} \cdot \hat{\mathbf{n}} ds$$

where $\hat{\mathbf{n}}$ is the unit normal pointing to the right of the curve (i.e. $\hat{\mathbf{T}}$ rotated $90°$ *clockwise*)
in coordinates $\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \int_C M dy - N dx = \int_C -N dx + M dy$ (to evaluate: same as before, since it is a line integral)
flux in space: flux of $\vec{F} = (P, Q, R)$ across a surface S in space is given by

$$\text{flux} = \iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_S \vec{F} \cdot d\vec{S}$$

where $\hat{\mathbf{n}}$ is a unit normal (orientation might be specified or left to you to choose).

- S is $z = f(x, y)$ the graph of $f(x, y) \implies \hat{\mathbf{n}} dS = \pm(-f_x, -f_y, 1) dx dy$
- S is parametrized by $\Phi(u, v) \implies \hat{\mathbf{n}} dS = \pm \Phi_u \times \Phi_v du dv$
- if we know that $\vec{N}$ is a normal vector to the surface S, then $\hat{\mathbf{n}} dS = \pm \dfrac{\vec{N}}{\vec{N} \cdot \hat{\mathbf{k}}} dA$ (e.g. slanted plane; level surface $g(x, y, z) = 0$ and $\vec{N} = \nabla g$.)

<table>
<tr><th></th><th>2D
$\vec{F} = (M, N)$</th><th>3D
$\vec{F} = (P, Q, R)$</th></tr>
<tr><td>work</td><td>Green's Theorem:
C = closed curve oriented counterclockwise enclosing region R

$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\mathrm{curl}\,\vec{F}) dA$

in coordinates:

$\int_C M dx + N dy = \iint_R (N_x - M_y) dA$

Application to area:
area (R) $= \frac{1}{2}\int_C x dy - y dx$

or
area (R) $= \int_C x dy$</td><td>Stokes' Theorem:
C = curve in space
S = any surface bounded by C
with compatible orientation

$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{\mathbf{n}} dS$

where $\nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Special case: if S is a closed surface (e.g. sphere, torus) then it has no boundary and the LHS of Stokes is 0.
In this case, $\iint_S \nabla \times \vec{F} = 0$
Note that this is true for vector curl of $\vec{F}$, not $\vec{F}$ itself!</td></tr>
<tr><td>flux</td><td>Green's theorem (normal form):
C and R as above
$\hat{\mathbf{n}}$ pointing outwards from R

$\int_C \vec{F} \cdot \hat{\mathbf{n}} ds = \iint_R (\mathrm{div}\,\vec{F}) dA$

in coordinates:
$\int_C M dy - N dx = \iint_R (M_x + N_y) dA$</td><td>Divergence theorem:
S = closed surface enclosing solid W
$\hat{\mathbf{n}}$ pointing outwards from R

$\iint_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iiint_W (\mathrm{div}\,\vec{F}) dV$

where $\mathrm{div}\,\vec{F} = P_x + Q_y + R_z$</td></tr>
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Have a nice winter break!