MATH 20E Lecture 8 - Tuesday, October 22, 2013

Work and line integrals

$W = (\text{force}) \cdot (\text{distance}) = \vec{F} \cdot \Delta\vec{r}$ for a small motion $\Delta\vec{r}$. Total work is obtained by summing these along a trajectory C : get a "line integral"

$$W = \int_C \vec{F} \cdot d\vec{r} \left(= \lim_{\Delta\vec{r}\to 0} \sum_i \vec{F} \cdot \Delta\vec{r}_i \right).$$

To evaluate the line integral, we observe C is parametrized by time t, with $a \le t \le b$ and give meaning to the notation $\int_C \vec{F} \cdot d\vec{r}$ by

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt.$$

Example: $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ and C is given by $x = t, y = t^2, 0 \le t \le 1$ (portion of parabola $y = x^2$ from $(0,0)$ to $(1,1)$). Then we substitute expressions in terms of t everywhere:

$$\vec{F} = (-y, x) = (-t^2, t), \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (1, 2t),$$

so $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_0^1 (-t^2, t) \cdot (1, 2t) dt = \int_0^1 t^2 dt = \frac{1}{3}$. (In the end things always reduce to a one-variable integral.)

New notation for line integral: $\vec{F} = (M, N)$, and $d\vec{r} = (dx, dy)$ (this is in fact a differential: if we divide both sides by dt we get the component formula for the velocity $d\vec{r}/dt$). So the line integral becomes

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy.$$

The notation is dangerous: this is not a sum of integrals w.r.t. x and y, but really a line integral along C. To evaluate one must express everything in terms of the chosen parameter.

In the above example, we have $x = t, y = t^2$, so $dx = dt, dy = 2tdt$; then

$$\int_C -y dx + x dy = \int_0^1 -t^2 dt + t(2t dt) = \int_0^1 t^2 dt = \frac{1}{3}.$$

(same calculation as before, using different notation).

In fact, the definition of the line integral does not involve the parametrization: so the result is the same no matter which parametrization we choose. For example we could choose to parametrize the parabola by $x = \sin\theta, y = \sin^2\theta, 0 \le \theta \le \pi/2$. Then we'd get $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \dots d\theta$ would be equivalent to the previous one under the substitution $t = \sin\theta$ and would again be equal to $1/3$. In practice we always choose the simplest parametrization!

Work in $3D$:

situation is similar to the one in the plane Given a vector field $\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = (P, Q, R)$ where P, Q, R are functions of x, y, z and a trajectory C in space we need to compute the work done by the vector field along C. This is given by the line integral

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt.$$

In coordinates: think of $d\vec{r} = (dx, dy, dz)$ and the line integral becomes

$$W = \int_C P dx + Q dy + R dz.$$

Example: $\vec{F} = (yz, xz, xy)$ and $C : x = t^3, y = t^2, z = t$ for $0 \le t \le 1$. Then $dx = 3t^2 dt, dy = 2t dt, dz = dt$ and substitute:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C yz dx + xz dy + xy dz = \int_0^1 6t^5 dt = 1.$$

(In general, express (x, y, z) in terms of a single parameter: 1 degree of freedom)

Geometric approach

Recall on trajectory C, velocity is $\dfrac{d\vec{r}}{dt} = \dfrac{ds}{dt}\hat{\mathbf{T}}$ where $s =$ arclength, $\hat{\mathbf{T}}=$ unit tangent vector to trajectory, $\frac{ds}{dt} =$ speed. So $d\vec{r} = \hat{\mathbf{T}} ds$ and

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\mathbf{T}} ds.$$

Sometimes the calculation is easier this way!

Example: $C =$ circle of radius a centered at origin; $\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ (points radially out). Then $\vec{F} \cdot \hat{\mathbf{T}} = 0$ because they are perpendicular (picture drawn), so $\int_C \vec{F} \cdot \hat{\mathbf{T}} ds = \int_C 0 ds = 0$.

Example: same C; $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ then $\vec{F}$ points in the same direction as $\hat{\mathbf{T}}$ so $\vec{F} \cdot \hat{\mathbf{T}} = \|\vec{F}\| = a$. Get that

$$\int_C \vec{F} \cdot \hat{\mathbf{T}} ds = \int_C a ds = a \int_C ds = a \cdot (\text{length of } C) = a(2\pi a) = 2\pi a^2.$$

We checked that we get the same answer if we compute using parametrization $x = a\cos\theta, y = a\sin\theta$.

More complicated trajectories; orientation

Example: $\vec{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$; $C = C_1 + C_2 + C_3$ enclosing sector of unit disk from 0 to $\pi/4$ (picture shown). Then work $= \int_C \vec{F} \cdot d\vec{r}$ is the sum of the work on each of C_1, C_2, C_3. So we need to compute $\int_{C_i} y dx + x dy$ for $i = 1, 2, 3$.

1) $C_1 : x$-axis from $(0, 0)$ to $(1, 0)$. Can do $x = t, y = 0, dx = dt, dy = 0, 0 \le t \le 1$. So

$\int_{C_1} y dx + x dy = \int_0^1 0 dt = 0$.
Equivalently, geometrically: along x-axis, $y = 0$ so $\vec{F} = x\hat{\mathbf{j}}$ while $\hat{\mathbf{T}} = \hat{\imath}$ (perpendicular). Therefore $\int_{C_1} \vec{F} \cdot \hat{\mathbf{T}} ds = 0$.

2) $C_2 : x = \cos\theta, y = \sin\theta, 0 \le \theta \le \pi/4$. Then $dx = -\sin\theta d\theta, dy = \cos\theta d\theta$. So

$$\int_{C_2} y dx + x dy = \int_0^{\pi/4} \sin\theta(-\sin\theta d\theta) + \cos\theta(\cos\theta d\theta) = \int_0^{pi/4} \cos(2\theta) d\theta = \left[\frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\pi/4} = \frac{1}{2}.$$

3) C_3 = line segment from $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ to $(0,0)$: could take $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t, y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t, 0 \le t \le 1$.

Easier: consider C_3 backwards (denoted C_3^-) which is parametrized by $x = y = t$ with $0 \le t \le \frac{1}{\sqrt{2}}$. Work along C_3^- is opposite of work along C_3.

$$\int_{C_3^-} ydx + xdy = \int_0^{1/\sqrt{2}} tdt + tdt = \left[t^2\right]_{t=0}^{t=1/\sqrt{2}} = \frac{1}{2} \implies \int_{C_3} ydx + xdy = -\frac{1}{2}.$$

Alternatively,

$$\int_{C_3} ydx + xdy = \int_{1/\sqrt{2}}^0 tdt + tdt = \left[t^2\right]_{t=1/\sqrt{2}}^{t=0} = -\frac{1}{2}.$$

Total work $= \int_{C_1} ydx + xdy + \int_{C_2} ydx + xdy + \int_{C_3} ydx + xdy = 0 + \frac{1}{2} - \frac{1}{2} = 0$.

Gradient fields

If $\vec{F}$ is a gradient field (i.e. $\vec{F} = \nabla f$ for some potential f) then we can use the **fundamental theorem of calculus for line integrals:**

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0) \text{ when } C \text{ runs from } P_0 \text{ to } P_1.}$$

Physical interpretation: the work done by a gradient field is given by the change in potential.

Proof (in 2 variables, but works in however many):

$$\int_C \nabla f \cdot d\vec{r} = \int_{t_0}^{t_1} \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right) dt = \int_{t_0}^{t_1} \frac{d}{dt}\left(f(x(t), y(t))\right) dt = \left[f(x(t), y(t))\right]_{t=t_0}^{t=t_1} = f(P_1) - f(P_0).$$

For instance, in the last example from Monday's lecture, we had $\vec{F} = (y, x) = \nabla f$ where $f(x, y) = xy$. (picture shown of C, $\vec{F}$ and level curves). We could compute $\int_{C_i}$ just by evaluating $f = xy$ at end points. Total work is 0 because we end where we started.

Consequences:

for a gradient field, we have:

- *Path independence:* if C_1, C_2 have same endpoints then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ (both equal to $f(P_1) - f(P_0)$ by the theorem). So the line integral $\int_C \nabla f \cdot d\vec{r}$ depends only on the end points, not on the actual trajectory.

- *Conservativeness:* if C is a closed loop then $\int_C \nabla f \cdot d\vec{r} = 0 (= f(P)?f(P))$. (e.g. in above example, $\int_C = 0 + 1/2 - 1/2 = 0$.)

WARNING: this is only for gradient fields!

Example: $\vec{F} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ is not a gradient field: as seen Monday, along C = circle of radius a counterclockwise ($\vec{F}$ is parallel to $\hat{\mathbf{T}}$), $\int_C \vec{F} \cdot d\vec{r} = 2\pi a^2$. Hence $\vec{F}$ is not conservative, and not a gradient field.

Physical interpretation

If the force field $\vec{F}$ is the gradient of a potential f, then work of $\vec{F}$ = change in value of potential. E.g.: 1) $\vec{F}$ = gravitational field, f = gravitational potential; 2) $\vec{F}$ = electrical field; f = electrical potential (voltage). (Actually physicists use the opposite sign convention, $\vec{F} = -\nabla f$). Conservativeness means that energy comes from change in potential f, so no energy can be extracted from motion along a closed trajectory (conservativeness = conservation of energy: the change in kinetic energy equals the work of the force equals the change in potential energy).

Note: path independence is equivalent to conservativeness by considering C_1, C_2 with same endpoints, $C = C_1 + C_2^-$ is a closed loop.

Surfaces in $\mathbb{R}^3$

1) Surface S is the graph of some function $z = f(x, y)$ over a region R of xy-plane: tangent plane at (x_0, y_0, z_0) to S where $z_0 = f(x_0, y_0)$ is given by

$$a(x - x_0) + b(y - y_0) = z - z_0 \text{ where } a = \frac{\partial f}{\partial x}(x_0, y_0) \text{ and } b = \frac{\partial f}{\partial y}(x_0, y_0).$$

Example: cannot remember what example I picked.

2) Surface S is given by the implicit equation $f(x, y, z) = c$ where c is a constant. We can think of this as the level surface $f = c$. The gradient vector $\nabla f(x_0, y_0, z_0)$ is normal to the tangent plane at (x_0, y_0, z_0). Equation of the plane is

$$\nabla f \cdot (x - x_0, y - y_0, z - z_0) = 0 \iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ where } (a, b, c) = \nabla f(x_0, y_0, z_0).$$

Example: tangent plane to hyperboloid $x^2 + y^2 - z^2 = 4$ (picture drawn) at $(2, 1, 1)$: gradient is $(2x, 2y, -2z) = (4, 2, -2)$; tangent plane is $4x + 2y - 2z = 8$. (Here we could also solve for $z = \pm\sqrt{x^2 + y^2 - 4}$ and use linear approximation formula, but in general we can't.)

MATH 20E Lecture 9 - Thursday, October 24, 2013

Surface area and tangent planes to surfaces

area of a surface is given by $\iint_{\text{surface}} dS$ where dS is the surface area element.

0) The xy coordinate plane: area element $dS = dA = dxdy$, normal vector $\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$, tangent plane = xy-plane itself.

1) Horizontal plane $z = a$: area element $dS = dxdy$, normal vector $\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$, tangent plane is $z = a$ plane itself.

2) Vertical plane $x = a$: area element $dS = dydz$, normal vector $\hat{\mathbf{n}} = \pm\hat{\mathbf{i}}$, tangent plane is $x = a$ plane itself.

3) If S is the graph of some function $z = f(x, y)$ over a region R of xy-plane: use x and y as variables. Contribution of a small piece of S to surface area?

Consider portion of S lying above a small rectangle $\Delta x \Delta y$ in xy-plane. In linear approximation it is a parallelogram. (picture shown)

The vertices are $(x, y, f(x, y)); (x+\Delta x, y, f(x+\Delta x, y)); (x, y+\Delta y, f(x, y+\Delta y))$; etc.

Linear approximation: $f(x+\Delta x, y) \approx f(x,y) + \Delta x f_x(x,y)$, and $f(x, y+\Delta y) \approx f(x,y) + \Delta y f_y(x,y)$. So the sides of the parallelogram are

$$(\Delta x, 0, f_x \Delta x) = (1, 0, f_x)\Delta x \text{ and } (0, \Delta y, f_y \Delta y) = (0, 1, f_y)\Delta y.$$

Thus the area of the parallelogram is equal to the norm of the cross product

$$(\Delta x, 0, f_x\Delta x)\times(0,\Delta y, f_y\Delta y) = (1,0,f_x)\times(0,1,f_y)\Delta x\Delta y = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \Delta x \Delta y = (-f_x, -f_y, 1)\Delta x \Delta y.$$

Therefore

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dx dy.$$

Note: the shadow R on xy-plane gives the bounds for the double integral

$$\text{area}(S) = \iint_R \sqrt{1+f_x^2+f_y^2}\, dx dy.$$

A normal vector to the surface is $(-f_x, -f_y, 1)$, hence $\hat{\mathbf{n}} = \pm \dfrac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$.

Example: area of cone $z = \sqrt{x^2+y^2}$ with $0 \le z \le 1$. The shadow on xy-plane is the unit disk $x^2 + y^2 \le 1$. The cone is therefore the graph of the function $f(x, y) = \sqrt{x^2+y^2}$ with $x^2+y^2 \le 1$.

The partial derivatives are $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$ so $\sqrt{1+f_x^2+f_y^2} = \sqrt{2}$. The area of the graph is

$$\iint_{x^2+y^2\le 1} \sqrt{2}\, dx dy = \sqrt{2}(\text{area of the unit disk}) = \pi\sqrt{2}.$$

4) Parametric surface $S: x = x(u,v), y = y(u,v), z = z(u,v)$ with $(u,v) \in R$ some region of the uv-plane.

Note: Since Φ_u and Φ_v are tangent vectors to the surface, they are contained in the tangent plane. So a normal vector to the tangent plane (and the surface) is $\Phi_u \times \Phi_v$, which is given by

$$\Phi_u \times \Phi_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \hat{\mathbf{k}} = \frac{\partial(y,z)}{\partial(u,v)}\hat{\mathbf{i}} - \frac{\partial(x,z)}{\partial(u,v)}\hat{\mathbf{i}} + \frac{\partial(x,y)}{\partial(u,v)}\hat{\mathbf{k}} \quad (1)$$

Example: S is parametrized by $x = u\cos v, y = u \sin v, z = u^2+v^2$. Find the tangent plane at $(u_0, v_0) = (1, 0)$.

We have $(x_0, y_0, z_0) = (1, 0, 1)$ is the point on S. The partial derivative vectors are $\Phi_u = (x_u, y_u, z_u) = (\cos v, \sin v, 2u)$ and $\Phi_v = (-u\sin v, u\cos v, 2v)$. At $(1, 0)$ they become $\Phi_u = (1, 0, 2)$ and $\Phi_v = (0, 1, 0)$. The normal vector is

$$\Phi_u \times \Phi_v = (1,0,2)\times(0,1,0) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{\mathbf{k}} = -2\hat{\mathbf{i}} + \hat{\mathbf{k}} = (-2, 0, 1).$$

The tangent plane is the plane with normal vector $\Phi_u \times \Phi_v = (-2, 0, 1)$ that passes through the point $(x_0, y_0, z_0) = (1, 0, 1)$, i.e. $-2(x-1) + (z-1) = 0$. Equation becomes $-2x + z + 1 = 0$.

For surface area, consider portion of S that is the image via Φ of a small rectangle $\Delta u \Delta v$ in uv-plane. In linear approximation it is a parallelogram (picture shown). The sides of the parallelogram are the vectors $\Phi_u \Delta u$ and $\Phi_v \Delta v$. The are of the parallelogram is given by the length/norm of the cross product of the two vectors. That is,

$$\Delta S = \|(\Phi_u \Delta u) \times (\Phi_v \Delta v)\| = \|\Phi_u \times \Phi_v\| \Delta u \Delta v$$

and therefore $dS = \|\Phi_u \times \Phi_v\| dudv$. Thus

$$\boxed{\text{area}(S) = \iint_R \|\Phi_u \times \Phi_v\| dudv.}$$

Since $\Phi_u \times \Phi_v$ is given by (1), we can compute its norm and get

$$\text{area}(S) = \iint_R \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, dudv.$$

Example: area of cone $z = \sqrt{x^2+y^2}$ with $0 \le z \le 1$. The shadow on xy-plane is the unit disk $x^2 + y^2 \le 1$. Parametrize by $x = r\cos\theta, y = r\sin\theta, z = r, 0 \le r \le 1, 0 \le \theta \le 2\pi$. Then $(x_r, y_r, z_r) = (\cos\theta, \sin\theta, 1)$ and $(x_\theta, y_\theta, z_\theta) = (-r\cos\theta, r\sin\theta, 0)$ and their cross-product is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 1 \\ -r\cos\theta & r\sin\theta & 0 \end{vmatrix} = \begin{vmatrix} \sin\theta & 1 \\ r\sin\theta & 0 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} \cos\theta & 1 \\ -r\cos\theta & 0 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} \cos\theta & \sin\theta \\ -r\cos\theta & r\sin\theta \end{vmatrix} \hat{\mathbf{k}} = -r\sin\theta \hat{\mathbf{i}} - r\cos\theta \hat{\mathbf{j}} + r\hat{\mathbf{k}}.$$

The norm is $r\sqrt{2}$ and the area of the cone is

$$\int_0^{2\pi} \int_0^1 r\sqrt{2} dr d\theta = \pi\sqrt{2}.$$

Note: Graphs are particular cases of parametric surfaces. That is, we can parametrize a graph using $\Phi(x,y) = (x, y, f(x,y))$ and then $\Phi_x = (1, 0, f_x), \Phi_y = (0, 1, f_y)$. Thus, we get

$$dS = \|\Phi_x \times \Phi_y\| dxdy = \|(-f_x, -f_y, 1)\| dxdy = \sqrt{1 + f_x^2 + f_y^2}\, dxdy.$$