MATH 20E Lecture 10 - Tuesday, October 29, 2013

More particular cases of parametric surfaces:

a. Sphere of radius a centered at origin: use φ, θ (substitute $\rho = a$ for evaluation);
Parametrization $\Phi : x = a\cos\theta\sin\varphi, y = a\sin\theta\sin\varphi, z = a\cos\varphi$ with $0 \le \varphi \le \pi, 0 \le \theta \le 2\pi$.
Then

$$\Phi_\varphi \times \Phi_\theta = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi & -a\sin\varphi \\ -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi & 0 \end{vmatrix} = (a^2\cos\theta\sin^2\varphi, a^2\sin\theta\sin^2\varphi, a^2\sin\varphi\cos\varphi).$$

Get $dS = a^2\sin\varphi d\varphi d\theta$ and $\hat{\mathbf{n}} = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) = \frac{1}{a}(x, y, z)$.

Example: surface area of the part of the unit sphere above the horizontal plane $z = \frac{1}{\sqrt{2}}$. (picture shown) Then $dS = \sin\varphi d\varphi d\theta$ and $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi/4$ (as $\pi/4$ is the value of φ at the intersection between the plane and the sphere). The surface area is given by

$$\iint_S dS = \int_0^{2\pi}\int_0^{\pi/4} \sin\varphi d\varphi d\theta = \int_0^{2\pi} [-\cos\varphi]_{\varphi=0}^{\varphi=\pi/4}\, d\theta = \int_0^{2\pi}\left(1 - \frac{1}{\sqrt{2}}\right) d\theta = \pi\left(2 - \sqrt{2}\right).$$

b. Cylinder of radius a centered on z-axis: use z, θ (substitute $r = a$ for evaluation).
Parametrization $\Phi : x = a\cos\theta, y = a\sin\theta, z = z$ with $0 \le \theta \le 2\pi$.

$$\Phi_\theta \times \Phi_z = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos\theta, a\sin\theta, 0) = (x, y, 0).$$

Get $dS = a dz d\theta$ and $\hat{\mathbf{n}} = \frac{1}{a}(x, y, 0)$ is radially out in horizontal directions away from z-axis.

More surfaces in $\mathbb{R}^3$:

5) Surface of revolution: S is obtained by taking the graph $y = f(x), a \le x \le b$ of a function of one variable and rotate around the x-axis (picture drawn).

$$\text{area}(S) = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\, dx$$

A small slice is a cylinder of height ds (arc length element) and with base a circle of radius $|f(x)|$. The length of the circle is $2\pi|f(x)|$ and $ds = \sqrt{1 + (f'(x))^2}\, dx$ (from MATH 20C).

Example: $f(x) = x, 0 \le x \le 1$. (picture drawn) Get cone of height 1, base circle of radius 1. Then the area is $2\pi \int_0^1 x\sqrt{2}\, dx = \pi\sqrt{2}$. (Same as last time).

6) Implicit surface $S : g(x, y, z) = 0$

For a slanted plane $ax + by + cz = d$, the normal vector is $\mathbf{N} = (a, b, c)$. Picture drawn. Surface element $\Delta S = ?$ Look at projection to xy-plane: $\Delta A = \Delta S|\cos\alpha| = (|\mathbf{N}\cdot\hat{\mathbf{k}}|/\|\mathbf{N}\|)\Delta S$ (where $\alpha =$ angle between slanted surface element and horizontal: projection shrinks one direction by factor $|\cos\alpha| = (|\mathbf{N}\cdot\hat{\mathbf{k}}|)/\|\mathbf{N}\|$, preserves the other).

Hence $dS = \dfrac{\|\mathbf{N}\|}{|\mathbf{N}\cdot\hat{\mathbf{k}}|}dxdy$. For a general implicit surface S given by equation $g(x,y,z)=0$ we use linear approximation. Normal vector to the surface is $\mathbf{N}=\nabla g$. Thus $dS = \dfrac{\|\nabla g\|}{|g_z|}dxdy$ and

$$\text{area}(S) = \iint_S dS = \iint_R \frac{\|\nabla g\|}{|g_z|}\,dxdy = \iint_R \sqrt{\frac{g_x^2+g_y^2+g_z^2}{g_z^2}}\,dxdy$$

where R is the shadow of S on the xy-plane.

Note: if S is vertical then the denominator is zero, can't project to xy-plane any more (but one could project e.g. to the xz-plane).

Green's Theorem

If C is a positively oriented (i.e. counterclockwise) closed curve enclosing a region R in the plane, then the work done by a vector field $\vec{F}=(M,N)$ – that is defined and differentiable everywhere in R – along C is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F})dA$$

where $\text{curl}(\vec{F}) = N_x - M_y$ is called the *scalar curl* of $\vec{F}$.

In coordinates:

$$\int_C Mdx + Ndy = \iint_R (N_x - M_y)dA.$$

Note: for a gradient field $\vec{F}=\nabla f=(f_x,f_y)$ we have

$$\text{curl}(\nabla f) = (f_y)_x - (f_x)_y = 0.$$

The scalar curl measures how far a vector field is from being a gradient.

Example (reduce a complicated line integral to an easy $\iint$): Let C = unit circle centered at $(2,0)$, counterclockwise. R = unit disk at $(2,0)$. Then

$$\int_C ye^{-x}dx + \left(\frac{1}{2}x^2 - e^{-x}\right)dy = \iint_R (N_x - M_y)dA = \iint_R (x+e^{-x}) - e^{-x}dA = \iint_R xdA.$$

Parametrize R in polar coordinates $x = 2 + r\cos\theta, y = r\sin\theta, 0\le\theta\le 2\pi, 0\le r\le 1$ and get

$$\iint_R xdA = \int_0^{2\pi}\int_0^1 (2+r\cos\theta)rdrd\theta = \int_0^{2\pi} 1 + \frac{\cos\theta}{3}d\theta = 2\pi.$$

(Note: direct calculation of the line integral would probably involve setting $x = 2+\cos\theta, y = \sin\theta$, but then we have exponential of trig functions and calculations get really complicated.)

Example (the condition for $\vec{F}$ to be defined and differentiable in R is essential): Let C = unit circle oriented counterclockwise and $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

Then

$$\operatorname{curl}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0.$$

Clicker question: is the work done by $\vec{F}$ along the closed curve C (i.e. $\int_C \vec{F}\cdot d\vec{r}$) equal to 0, positive or negative?

We can compute $\int_C \vec{F}\cdot d\vec{r}$ directly. Parametrize C as usual by $x = \cos\theta, y = \sin\theta$ with $0 \le \theta \le 2\pi$. Then $dx = -\sin\theta d\theta$ and $dy = \cos\theta d\theta$. Therefore

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \frac{-ydx + xdy}{x^2+y^2} = \int_0^{2\pi} \frac{-\sin\theta(-\sin\theta d\theta) + \cos\theta(\cos\theta d\theta)}{\cos^2\theta + \sin^2\theta} = \int_0^{2\pi} d\theta = 2\pi \neq 0.$$

Green's theorem *does not apply* here because the region R enclosed by C is the unit disk and $\vec{F}$ is not defined at the origin.

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More about Green's Theorem

Example: $\vec{F} = (-y, x)$; have seen $\operatorname{curl}\vec{F} = 2$. Then for any closed curve C that encloses region R and is oriented counterclockwise. Plugging into $\int_C \vec{F}d\vec{r} = \iint_R (\operatorname{curl}\vec{F})dA = 2\iint_R dA$ get

$$\boxed{\text{area}(R) = \frac{1}{2}\int_C xdy - ydx}$$

Example (reduce a complicated area integral to an easy line integral): example 2, section 8.1.

Example: $\vec{G} = (0, x)$; have seen $\operatorname{curl}\vec{G} = 1$. Then for any closed curve C that encloses region R and is oriented counterclockwise. Plugging into $\int_C \vec{G}d\vec{r} = \iint_R (\operatorname{curl}\vec{G})dA = \iint_R dA$ get

$$\boxed{\text{area}(R) = \int_C xdy}$$

Review for Midterm

Topics: functions of several variables, partial derivatives, chain rule, approximation formula, tangent planes; derivative matrix;
integration in several variables, change of variables
surfaces in space: normal vector, area element, surface area; parametrization
vector fields: flow lines, work in plane and space, Fundamental Theorem of Calculus, Green't theorem

Discussed problems 4 (partial derivatives, gradient, tangent plane and approximation formula), 5 (chain rule), 11 (general changes of variables) from the study guide.

Recall for general changes of variables: $u = u(x, y), v = v(x, y)$. The Jacobian is $J = \frac{\partial(u,v)}{\partial(x,y)} \stackrel{\text{def}}{=} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Then $dudv = |J|dxdy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy$ (absolute value because area is the absolute value of the determinant).

We ran out of time at this point, so did not go over change of variables in 3D, surfaces or vector fields.